PHYSICAL REVIEW LETTERS

VOLUME 58 | 22 JUNE 1987 | NUMBER 25

Metamorphosis and Duality between Quantum Systems

Fernando Cabral and Jason A. C. Gallas

Departamento de Física, Universidade Federal de Santa Catarina, 88049 Florianopólis, Brazil

(Received 29 December 1986)

We show that a duality transformation (connecting integrable Hamiltonian systems) recently discovered by Hietarinta and co-workers in two dimensions at a classical level can be directly established in N dimensions from a quantum formulation of the problem. Further, we show duality to be valid for a much larger class of dynamical systems and to be not necessarily unique. We speculate that this nonuniqueness could be a characteristic of separable systems.

PACS numbers: 03.20.+i, 02.30.+g

A number of recent papers have been devoted to the investigation of certain *duality* properties between pairs of Hamiltonian systems. In this context the term duality is used to mean that once a given property of one of the Hamiltonians is established, the same property is expected to show up in the dual partner. The existence of such a duality between Hamiltonians is obviously of enormous interest: The discovery of one integrable dynamical system (usually a difficult task, specially in higher dimensions) would automatically imply the existence of another integrable system, nonintegrability of one of the partners implies the nonintegrability of the other, chaos in one implies chaos in the other, etc.

The first example of what we now like to call duality is the quite early discovery[1] of a transformation connecting the radial equations of the harmonic oscillator and the Coulomb potential. Coincidentally (or not!) these potentials are the only ones for which all bound orbits are closed (Bertrand's theorem) and ellipses.[2] On a different but equivalent language, these potentials are the only ones for which, besides energy and angular momentum, a further constant of motion (the Runge-Lenz vector) exists. Although the harmonic oscillator and the Coulomb potentials are three dimensional, they are separable and, therefore, the duality as established in Ref. 1 is a property of two one-dimensional (in particular, radial) equations. Over the years the existence of an *equivalence* between the harmonic oscillator and the Coulomb potential has been rediscovered by Bergmann and Frishman,[3] Dulock and McIntosh,[4] Talman,[5] and Rockmore,[6] at least.

The next interesting example of duality was obtained by Feldman, Fulton, and Devoto,[7] Quigg and Rosner,[8] and Collas.[9] These authors established that to every radial equation containing a potential r^q there was another dual equation containing a potential r^p, with p and q connected by the equation

$$p+q+\tfrac{1}{2}pq=0. \tag{1}$$

This very symmetrical equation is particularly beautiful because it connects the bound-state spectrum of a confining potential r^q, $q>0$ with that of a potential r^p, $-2<p<0$, which is singular at the origin. In particular, the harmonic oscillator ($q=2$) is easily seen to be dual of the hydrogen atom ($p=-1$) and vice versa. The important point to realize here is that for one-dimensional systems Eq. (1) was established in a quantum as well as in a classical way: through the semiclassical WKB approximation,[7] through transformations on the Schrödinger equation,[8] and through local diffeomorphisms connecting classical Hamiltonians of different potentials.[9]

Recently, and apparently unaware of the aforementioned 1D results, Hietarinta[10] used Painlevé analysis to conclude that two bidimensional Hamiltonians of the type

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+Cx^{\alpha+2}+x^\alpha y^2 \tag{2}$$

with parameters α_1 and α_2 are dual if α_1 and α_2 obey

$$\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_1\alpha_2=0. \tag{3}$$

In this way he was able to show that several previously uncorrelated pairs of integrable two-dimensional dynamical systems were indeed dual to each other. Using Eq. (3) he was also able to find new "missing" integrable systems. Later, Hietarinta *et al.*[11] found a noncanonical transformation which explained at a classical level the observed duality. Such transformation interchanged energy into coupling constant and vice versa while preserving important properties such as integrability. Using *c*-number representation for quantum operators along with Moyal brackets for commutators, the authors of Refs. 10 and 11 were also able to gain some information about the quantum integrability of the dual transformation. For example, while for $\alpha=1$, 0, -4, and -6 the Hamiltonian (2) is classical as well as quantum integrable, for $\alpha=-\frac{2}{3}$ it is found to be classical but not quantum integrable.[10] However, in this particular case, Hietarinta discovered that it was possible to "restore" quantum integrability through a *suitable deformation* of the potential by an extra term proportional to $\hbar^2$ so that

$$V(x,y)=Cx^{4/3}+x^{-2/3}y^2-\tfrac{5}{72}\hbar^2x^{-2}. \tag{4}$$

This "net quantum correction" that had to be added to the potential was later[11] found to be related to a Schwarzian derivative recently discussed by Weiss[12] in connection with properties of integrable systems.

The purpose of the present paper is to show that the duality as well as the coupling-constant metamorphosis discovered by Hietarinta and co-workers[10,11] at a classical level for 2D systems can be directly obtained from a quantum formulation of the problem. Thus, instead of the *ad hoc* introduction of deformations proportional to $\hbar^2$ in the potential we derive these "corrections" from transformations of the Schrödinger equation. Further, we show duality to hold for systems of arbitrary dimensionality and to be valid for a much larger class of dynamical systems than previously thought. We conclude by showing that duality may not be unique in the sense that in some cases it is possible to find more than one dual partner for a given quantum system even when the potential has only one free coupling constant. We speculate that this "abundance of duality" could be a characteristic of separable systems. Before starting, it is perhaps important to stress that quantum integrability is not a trivial consequence of classical integrability.[13]

The similarity between Eqs. (1) and (3) suggests that we try the same change of dependent and independent variables as in the one-dimensional case[8]: It does not work. The noncanonical transformation discussed in Ref. 11 suggests that we try a change of the time scale in the time-dependent Schrödinger equation, but this does not work either. What works is a combination of a Fourier transform (of some of the variables) after a convenient change of the (non-Fourier-transformed) variables. It is important to note that the noncanonical transformation used in Ref. 11 is more than just a change of time scale. The fundamental step in it is the interchange of y and P_y. The quantum equivalent of this step is the Fourier transform.

Let a dynamical system be described by a potential

$$V(x,y)=f(x)-Cx^s-\epsilon x^q+Dx^qy^2+L^2/x^2, \tag{5a}$$

with the corresponding bidimensional Schrödinger equation given by

$$-A(\psi_{xx}+\psi_{yy})+V(x,y)\psi=E\psi. \tag{5b}$$

The parameter $A\equiv\hbar^2/(2\mu)$ defines the energy scale while $f(x)$ is an arbitrary function of its argument. Let us now transform x and ψ in Eq. (5b) according to the prescription

$$\xi=x^{-q/p},\quad \psi(x,y)=\xi^{-(1+p/q)/2}\tilde{\phi}(\xi,y), \tag{6}$$

and, subsequently, Fourier transform the y variable in the equation. This leads to

$$A\frac{q^2}{p^2}\xi^{2(1+p/q)}\frac{\partial^2\phi}{\partial\xi^2}+D\hbar^2\xi^p\frac{\partial^2\phi}{\partial\eta^2}+\left\{E-\left[F(\xi)\xi^{-p}-\epsilon\xi^{-p}-C\xi^{-sp/q}-\frac{A}{4}\left(\frac{q^2}{p^2}-1\right)\xi^{2p/q}+\frac{A}{\hbar^2}\eta^2+L^2\xi^{2p/q}\right]\right\}\phi=0, \tag{7}$$

where now $\phi=\phi(\xi,\eta)$ and $F(\xi)=f(\xi^{-p/q})\xi^p$. This equation can be brought to the form of Eq. (5) by our changing $\eta\to k\eta$ with $k^2=Aq^2/(D\hbar^2p^2)$ and choosing p such that

$$p+q+\tfrac{1}{2}pq=0. \tag{8}$$

Equation (7) reduces to (writing η instead of $k\eta$)

$$-A(\phi_{\xi\xi}+\phi_{\eta\eta})+V(\xi,\eta)\phi=p^2\epsilon\phi/q^2, \tag{9a}$$

where

$$V(\xi,\eta)=\frac{p^2}{q^2}\left\{F(\xi)-E\xi^p-C\xi^{p-sp/q}+D\frac{p^2}{q^2}\xi^p\eta^2+\left[L^2-\tfrac{1}{4}A\left(\frac{q^2}{p^2}-1\right)\right]\xi^{-2}\right\}. \tag{9b}$$

The Schrödinger equations (5) and (9) are dual to each other. The parameter p in Eq. (9b) is connected with q of Eq. (5a) through relation (8) which is identical to Eq. (3) that was found classically by Hietarinta.[10] Equation (9b) clearly shows why for $q=-\frac{2}{3}$ a $-\frac{5}{72}\hbar^2x^{-2}$ deformation was needed in Eq. (4) for the system to be quantum integrable. For a given q the term proportional to ξ^{-2} will be absent in Eq. (9b) whenever we start from a potential having

$$L^2=\tfrac{1}{4}A(q^2/p^2-1)=\tfrac{1}{16}Aq(q+4). \tag{10}$$

For $q=-\frac{2}{3}$ one obtains $L^2=-\frac{5}{36}A=-\frac{5}{72}\hbar^2/\mu$, in perfect agreement with the empirical finding of Hietarinta.[10] Further, Eq. (10) shows that $q=-\frac{2}{3}$ is not the only case requiring a "quantum deformation" and, in particular, allows one to predict correction terms for all cases listed in Table I of Ref. 10.

Comparing Eq. (5) with Eq. (9) one sees that the coupling constant ϵ becomes the energy of the dual equation while E (the energy of the starting equation) becomes one of the coupling constants in the dual potential, exactly as in the noncanonical classical case.[11] Further, note that Eq. (5a) contains a term $(Dy^2-\epsilon)x^q$ while Eq. (9b) contains $(p^2/q^2)(p^2D\eta^2/q^2-E)\xi^p$ and that the simultaneous occurrence of these terms is critical for quantum duality to exist.

Another interesting feature of duality is the "universality" of Eq. (8), i.e., the validity of Eq. (8) to connect dual systems in N-dimensional spaces, $N=1,2,\ldots$. For $N=1$ we already saw that duality can be established classically as well as quantically. In fact, in this case duality is much simpler to establish since for $D=0$ the Schrödinger equation (5) separates and no Fourier transformation is needed. We already mentioned the duality between the Coulomb and the harmonic oscillator. Another interesting 1D dual pair should be that associated with the linear potential. The linear potential $(p=1)$ has well-known closed-form solutions (in terms of Airy functions). We should therefore expect closed-form solutions to exist for the potential $r^{-2/3}$.

From the above discussions one realizes that duality is also valid for three-dimensional potentials of the generic type

$$V(x,y,z)=f(x)+(az^2+by^2-\epsilon)x^q, \tag{11}$$

where a and b are coupling constants, and that this result can be trivially extended to arbitrary dimensions.

Let us now investigate quantum duality for potentials of a different type than those so far discussed. As we have seen, the transformations defined by Eq. (6) together with suitable Fourier transformations and scalings can lead to dual systems whenever free coupling constants appear in the potential. Our next objective is to show through a 2D example that *duality is not necessarily unique* and that, in fact, we may have families of potentials sharing dual-type properties. To this end let us take now a 2D Schrödinger equation in polar coordinates with a generic potential of the form

$$V(r,\theta)=\lambda r^q+f(r,\theta). \tag{12}$$

In these coordinates the Schrödinger equation is given by

$$-A\nabla^2\phi+V(r,\theta)\phi=-A\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]+V(r,\theta)\phi=E\phi, \tag{13}$$

which, upon changing $\phi(r,\theta)=r^{-1/2}\psi(r,\theta)$, can be more conveniently written as

$$-A\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)+\lambda r^q\psi+f(r,\theta)\psi-\frac{A}{4r^2}\psi=E\psi. \tag{14}$$

Substituting $\rho=r^{-q/p}$, $\tilde\theta=q\theta/p$, and $\psi(r,\theta)=\rho^{-(1+p/q)/2}\tilde\psi(\rho,\tilde\theta)$, one obtains the dual equation

$$-A\left(\frac{\partial^2\tilde\psi}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2\tilde\psi}{\partial\tilde\theta^2}\right)+\tilde\lambda\rho^p\tilde\psi+F(\rho,\tilde\theta)\tilde\psi-\frac{A}{4\rho^2}\tilde\psi=\tilde E\tilde\psi, \tag{15}$$

where

$$\tilde\lambda=-p^2E/q^2,\quad \tilde E=-p^2\lambda/q^2,\quad F(\rho,\tilde\theta)=f(\rho^{-p/q},\tilde\theta p/q)\rho^p p^2/q^2, \tag{16}$$

with p and q connected by the universal equation (8). Let us now analyze the (separable) axially symmetric potential

$$V(x,y)=\tfrac{1}{24}\lambda(x^2+y^2)^2=\tfrac{1}{24}\lambda r^4, \tag{17}$$

already investigated by Hietarinta *et al.*[11] From Eq. (8) we obtain a $\rho^{-4/3}$ dual for this potential. Besides connecting $V(x,y)=\lambda(x^2+y^2)^2$ with $V(\tilde x,\tilde y)=\tilde\lambda(\tilde x^2+\tilde y^2)^{-4/3}$ Eqs. (14) and (15) share the property of transforming $-\frac{1}{4}Ar^{-2}$ into $-\frac{1}{4}A\rho^{-2}$ and vice versa, therefore bypassing any need for quantum corrections. This property is easily explained by Eq. (9b) which shows $L^2=-\frac{1}{4}A$ to be the solution of the equation $L^2=\tilde L^2$ where

$$\tilde L^2=p^2[L^2-\tfrac{1}{4}A(q^2-p^2)/p^2]/q^2. \tag{18}$$

In Cartesian coordinates the Schrödinger equation for the potential (17) is

$$-A(\psi_{xx}+\psi_{yy})+\tfrac{1}{24}\lambda(x^2+y^2)^2\psi=E\psi, \quad (19)$$

which can also be written as

$$-A\left[\frac{i4\omega}{\sigma\sqrt{2}}\left(v^2\frac{\partial\phi}{\partial u}+u^2\frac{\partial\phi}{\partial v}\right)+2\sigma^2\frac{\partial^2\phi}{\partial u\,\partial v}\right]=E\phi, \quad (20)$$

where

$$\psi=\phi\exp[i\omega(\tfrac{1}{3}x^3-xy^2)],\quad \omega^2=-\lambda/24A, \quad (21)$$

$$u=\sigma(x+iy)/\sqrt{2}, \quad (22a)$$

$$v=\sigma(x-iy)/\sqrt{2}. \quad (22b)$$

Choosing now $\sigma=4\omega\hbar/\sqrt{2}$ and Fourier transforming Eq. (20) we obtain

$$-A\left[U\frac{\partial^2\tilde\phi}{\partial V^2}+V\frac{\partial^2\tilde\phi}{\partial U^2}\right]-\tfrac{2}{3}\lambda UV\tilde\phi=E\tilde\phi, \quad (23)$$

which is of the generic type

$$-A\left[u^p\frac{\partial^2\phi}{\partial v^2}+v^r\frac{\partial^2\phi}{\partial u^2}\right]-Cu^pv^r\phi=E\phi. \quad (24)$$

Introducing

$$\phi(u,v)=x^\alpha y^\beta\psi(x,y),\quad x=u^{-a},\quad y=v^{-b},$$

$$\alpha=-(1+a)/2a,\quad \beta=-(1+b)/2b,\quad q=p/a,$$

and

$$s=r/b,$$

after a convenient scaling of x and y, Eq. (24) is transformed into

$$-A\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{p^2-q^2}{4p^2x^2}\psi+\frac{r^2-s^2}{4r^2y^2}\psi\right]-E\left(\frac{p}{q}\right)^q\left(\frac{r}{s}\right)^s x^qy^s\psi=C\psi, \quad (25)$$

with p and q connected by Eq. (8) and r and s obeying $r+s+\frac{1}{2}rs=0$. For the particular case $p=r=1$ of Eq. (23) one finds $q=s=-\frac{2}{3}$ and, therefore,

$$-A\left(\frac{p^2-q^2}{4p^2x^2}+\frac{r^2-s^2}{4r^2y^2}\right)=-\frac{5}{72}\frac{\hbar^2}{\mu}\left(\frac{1}{x^2}+\frac{1}{y^2}\right). \quad (26)$$

In this case Eq. (25) is reduced to the Schrödinger equation for the $(xy)^{-2/3}$ Fokas-Lagestrom potential and is the quantum version of the classical results of Hietarinta *et al.*[11] From the above calculations one sees that the potential

$$V=\lambda(x^2+y^2)^2 \quad (27)$$

has two duals, namely,

$$V_1=\tilde\lambda(x^2+y^2)^{-4/3}, \quad (28)$$

$$V_2=\tilde{\tilde\lambda}(xy)^{-2/3}-\frac{5}{72}\frac{\hbar^2}{\mu}\left(\frac{1}{x^2}+\frac{1}{y^2}\right). \quad (29)$$

Our analysis indicates the nonuniqueness of the duality to be related to the separability of the problem, but the extension of any eventual relation remains an open question. Based on the above example we speculate that any "abundance of duality" could be an indicative property of separable systems.

In summary, we established a duality between quantum systems, valid in N dimensions. For two-dimensional systems and in the limit $\hbar\to 0$ our results agree with the classical duality recently obtained by Hietarinta and co-workers.[10,11] Further, we have shown duality to be valid for a much larger class of dynamical systems and to be not necessarily unique.

The work of our group is partially supported by the Brazilian research council Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). One of us (J.A.C.G.) is a CNPq research fellow.

[1]E. Schrödinger, Proc. Roy. Irish Acad. **46**, 183 (1941). See also J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641 (1940); T. Levi-Cività, "Sur la résolution qualitative du problème restreint des trois corps," in *Opere Mathematiche* (Zanichelli, Bologna, 1956), Vol. 2; E. L. Steifel and G. Scheifele, *Linear and Celestial Mechanics* (Springer, Berlin, 1971).

[2]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980); V. I. Arnold *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).

[3]D. Bergmann and Y. Frishman, J. Math. Phys. **6**, 1855 (1965).

[4]V. A. Dulock and H. V. McIntosh, Pac. J. Math. **19**, 39 (1966).

[5]J. D. Talman, *Special functions: A Group Theoretic Approach* (W.A. Benjamin, New York, 1968).

[6]R. Rockmore, Am. J. Phys. **43**, 29 (1975).

[7]G. Feldman, T. Fulton, and A. Devoto, Nucl. Phys. **B154**, 441 (1979).

[8]C. Quigg and J. L. Rosner, Phys. Rep. **56**, 167 (1979).

[9]P. Collas, J. Math. Phys. **22**, 2512 (1981).

[10]J. Hietarinta, Phys. Rev. A **28**, 3670 (1983).

[11]J. Hietarinta, B. Grammaticos, B. Dorizzi, and A. Ramani, Phys. Rev. Lett. **53**, 1707 (1984).

[12]J. Weiss, J. Math. Phys. **24**, 1405 (1983).

[13]J. Hietarinta, Phys. Lett. **93A**, 55 (1982), and J. Math. Phys. **25**, 1833 (1984); J. Hietarinta, B. Dorizzi, B. Grammaticos, and A. Ramani, C.R. Acad. Sci. Ser. 2 **297**, 791 (1983).