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Noisy collective behaviour in deterministic cellular automata

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We investigate cellular automata in four and five dimensions for which Chaté and Manneville recently have found nontrivial collective behaviour. More precisely, though being fully deterministic, the average magnetization seems to be periodic respectively quasiperiodic, with superimposed noise whose amplitude decreases with system size. We confirm this behaviour on very large systems and over very large times. We analyse in detail the statistical properties of the "noise". Systems on small lattices and/or subject to additional external noise are metastable. Arguments by Grinstein et al. suggest that in the periodic case the infinite deterministic systems should be metastable too. These arguments are generalized to quasiperiodic systems. We find evidence that they do indeed apply, but we find no direct evidence for metastability of large systems.

1. Introduction

Recently, much attention has been paid to extended homogeneous systems far from equilibrium. In particular, the question arose whether such systems can show global collective behaviour when governed by short range interactions and subject to noise. Naively, one could expect that the noise would affect different parts of the system in different ways, whence distant parts of the system would run out of phase. Then, any globally averaged observable would not show any non-trivial time dependence in the limit of infinite system size.

In ref. [1], this argument was made sharp for chaotic systems such as coupled map lattices. There, amplitudes of small fluctuations blow up exponentially, but they spread only with fixed velocity. Thus, it is impossible to have any mechanisms which could synchronize the motion over large distances: at sufficiently large distance, any suppression of a desynchronizing fluctuation would come too late.

For periodic and quasiperiodic collective behaviour, the situation is more subtle. Yet, in refs. [2, 3] convincing arguments were given that no periodic behaviour with period $\geqslant 3$ should be expected in isotropic systems. These arguments are reviewed in section 3, where also the generalization to quasiperiodic systems is given.

Very surprising in view of this were thus numerical observations by Chaté and Manneville [4], who found a five-dimensional deterministic cellular automaton (CA) which seemed to show noisy quasiperiodic behaviour. Later, Chaté and Manneville ref. [5] found noisy quasiperiodic and periodic behaviour also in four-dimensional CA's. These automata and some basic results are described in more detail in section 2.

A priori, there are several possibilities how the observations of refs. [4] and [2, 3] could be reconciled:

– While Bennett et al. assume the noise to be random, this is obviously not true in the CA which are strictly deterministic and not even chaotic since phase space is discrete.

– The arguments of Bennett et al. hold only for isotropic systems, thus they might not hold on a regular lattice.

– These arguments use in a crucial way the notion of a phase boundary or Bloch wall. The fact that (quasi-)periodicity is only observed in $\geqslant 4$ dimensions might suggest that this notion has to be taken with care in high dimensions. This would indeed be in line with the experience from critical phenomena.

– Finally, it might be that the (quasi-)periodic states observed by Chaté and Manneville are only metastable. This would indeed be the easiest way out, since metastable periodic states are not forbidden by the arguments of Bennett et al. It would of course diminish somewhat the interest in these states, but not very much since the lifetimes would have to be extremely large.

In sections 3–6 we shall discuss each of these possibilities in more detail. We find no evidence that any of them can explain the findings of Chaté and Manneville.

In section 3 we shall see that the arguments of Bennett et al. can be extended to quasiperiodic behaviour. Indeed, in agreement with the naive view that quasiperiodicity is "between" chaos and periodicity we find that the argument there is even stronger than for periodic behaviour: while the latter could arise from anisotropy on small length scales, only anisotropy which survives also in the long wavelength limit could explain quasiperiodic behaviour. Also, we do not need sharp phase boundaries (Bloch walls) for quasiperiodic behaviour. Instead, it is sufficient if phases can be defined locally in the weak sense that systems can support phase gradients. This is verified numerically for the 5-dimensional Chaté–Manneville CA.

In section 4 we show that the "noise" superimposed on the collective motion behaves indeed exactly like bona fide stochastic noise, and that its strength scales with system sizes exactly as predicted by naive arguments.

In section 5 we study metastability in finite systems and in systems subject to external noise. In the limits of large system size respectively small noise level, the lifetimes are described by scaling laws which again could have been guessed easily, but which are hard to reconcile with the nucleation picture suggested by ref. [2].

Finally, anisotropy is studied in section 6, where we show that it most likely does not explain periodic oscillation.

The paper finishes with conclusions in section 7.

2. The Chaté–Manneville automata

All three CA studied in this paper are defined on d-dimensional hypercubic lattices ($d = 4, 5$) and have 2 states per site, denoted as $s_i = 0, 1$. They are *totalistic* in the sense of ref. [6], i.e. the "spin" at site i and time $t+1$ is given by the *sum* of all spins in the von Neumann neighbourhood of i at time t,

$$s_i^{t+1} = f\Big(\sum_{\langle i,j\rangle} s_j^t\Big) . \tag{1}$$

Here, the symbol $\langle i, j\rangle$ means that site j is in the von Neumann neighbourhood of i, i.e. either $j = i$ or $j = i \pm e_n$, where e_n is the nth unit vector on the lattice. Since the sum in eq. (1) can take values from 0 to $2d+1$, the CA is uniquely defined by giving $f(x)$ for $x = 0, \dots, 2d+1$. Out of this class which is still extremely large for large d, only rules are considered for which (similar to Conway's Game of Life [7])

$$s_i^{t+1} = \begin{cases} 1 & \text{for } k \le \sum_{\langle i,j\rangle} s_j^t \le l , \\ 0 & \text{otherwise} . \end{cases} \tag{2}$$

More precisely, we consider the rules with $k = 5$, $l = 9$ in $d = 5$, with $k = 3$, $l = 8$ in $d = 4$, and with $k = 4$, $l = 9$ in $d = 4$. They are denoted as $R^5_{5,9}$, $R^4_{3,8}$ and $R^4_{4,8}$. Rule $R^5_{5,9}$ was studied in ref. [4], rules $R^4_{3,8}$ and $R^4_{4,8}$ in ref. [5]. In addition to these deterministic CA, in ref. [4] also probabilistic rules were considered. They show similar behaviour, but we shall not discuss them here. In all cases (with one exception mentioned below), we used periodic boundary conditions on hypercubes of size L^d.

If we start with random initial configurations where $s_i = 0$ and $s_i = 1$ with equal probability, $R^4_{3,8}$ runs very fast (after less than 100 iterations) into a noisy period-3 orbit, as measured by the average value of s (or "magnetisation"),

$$m_t = \frac{1}{L^d} \sum_i s_i^t , \tag{3}$$

while $R^4_{4,8}$ and $R^5_{5,9}$ lead after similarly short transients to quasiperiodic behaviour. This is illustrated in figs. 1–3. In these figures m_{t+1} is plotted against m_t. While fig. 1 shows three diffuse clouds indicating a noisy period 3, figs. 2 and 3 show fuzzy closed loops indicative of quasiperiodicity. In all cases the fuzziness decreases with system size L, suggesting that the "noise" might disappear in the limit $L \to \infty$.

In order to exclude that these states are only metastable with not too large lifetimes, we performed very long iterations: up to 500 000 time steps for $R^5_{5,9}$ on a lattice with $L = 16$, and 100 000 steps for the 4-dimensional rules on lattices with $L = 32$ and $L = 64$ (these large simulations were possible by using the Connection Machines of Wuppertal University and of GMD; smaller lattices were simulated on a Cray and on workstations). The resulting plots of m_{t+1} against m_t were indistinguishable from those in figs. 1–3.

In spite of the fact that one of the rules gives periodic behaviour and the other ones lead to quasiperiodicity, all three rules are very similar. This is seen most clearly by the fact that the winding numbers in the latter are very close to $\frac{2}{3}$. The winding number w is defined by first defining an angle variable for the position on the circle as

$$\phi_t = \arctan \frac{m_{t+1} - m^*}{m_t - m^*} \tag{4}$$

with (m^*, m^*) a point in the interior of the loops in the m_t–m_{t+1} plane (we used $m^* = 0.705$ for $R^5_{5,9}$ and $m^* = 0.76$ for $R^4_{4,8}$) and with $\phi_{t+1} - \phi_t \in [0, 2\pi)$, and writing then

$$w = \lim_{t \to \infty} \frac{\phi_t}{2\pi t} . \tag{5}$$

As for any quasiperiodic motion characterised by a single winding number, there exists a conjugacy (a smooth invertible map $\psi = G(\phi)$) which maps ϕ onto a variable ψ whose evolution is a pure shift,

$$\psi_{t+1} = \psi_t + w . \tag{6}$$

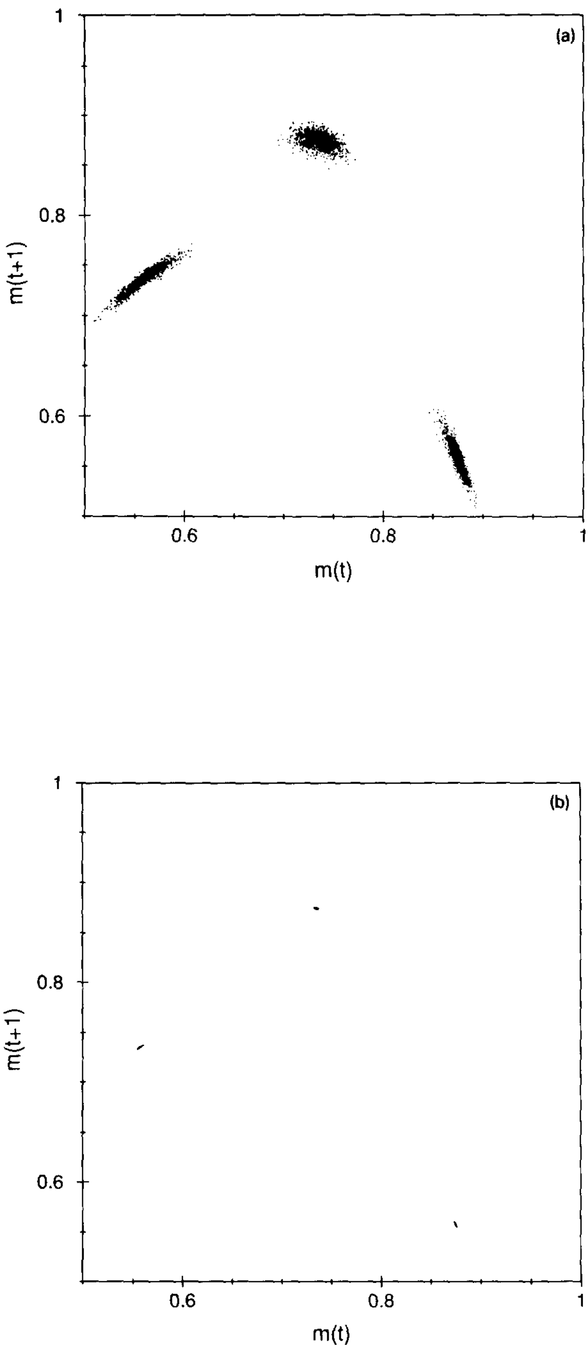

Fig. 1. m_{t+1} versus m_t for rule $R^4_{3,8}$ for $1000 < t < 5000$: (a) for $L = 16$; (b) for $L = 64$.

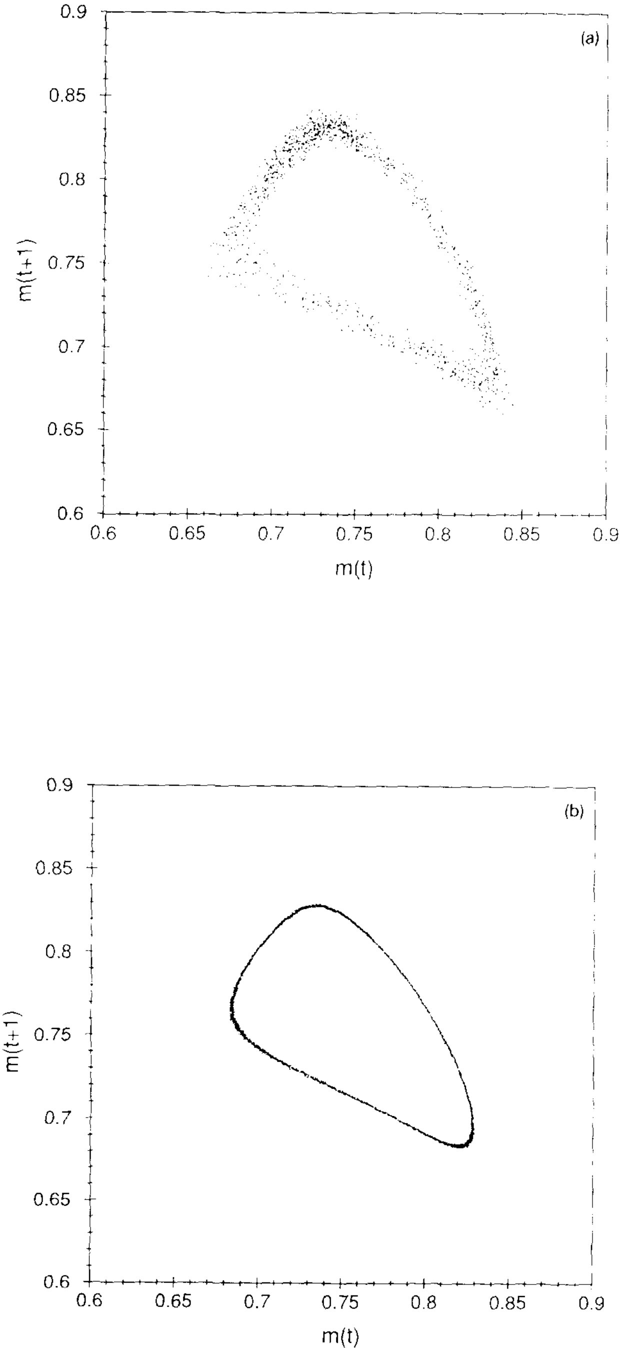

Fig. 2. Same as fig. 1, but for rule $R^{+}_{4,8}$.

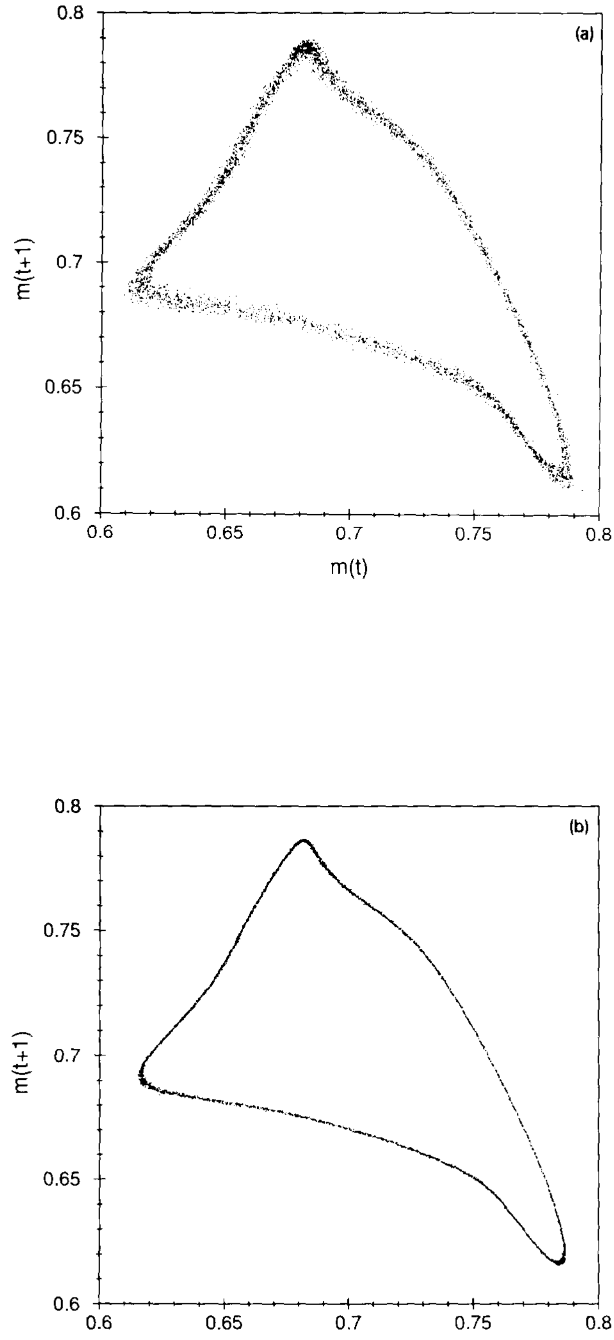

Fig. 3. m_{t+1} against m_t for rule $R^5_{5,9}$ for $1000 < t < 5000$: (a) for $L = 16$; (b) for $L = 32$.

The most precise determination of w would presumably involve an explicit construction of this map. In a somewhat simpler approach, we used the well known fact that optimal delay coordinates are such that successive points have $\phi_{t+\Delta t} - \phi_t \approx \pm\frac{1}{2}\pi$ [8]. This is not true for $\Delta t = 1$ as in figs. 1–3 and in eq. (4), but it happens to be true for $\Delta t = 5$, i.e. if one uses m_{t+5} versus m_t. With this embedding, we computed the average number of windings during T iterations,

$$W_T = \frac{1}{2\pi} \langle \phi_{t+T} - \phi_t \rangle , \tag{7}$$

and their fluctuation

$$\Delta W_T = \frac{1}{2\pi} [\langle (\phi_{t+T} - \phi_t)^2 \rangle - 4\pi^2 W_T^2]^{1/2} . \tag{8}$$

ΔW_T gets essentially two contributions: for large T, statistical fluctuations should dominate if the jitter seen in figs. 2 and 3 is equivalent to random noise. They should give $\Delta W_T \propto \sqrt{T}$. For small T, on the other hand, we have nonrandom fluctuations which arise from the fact that the winding is not uniform in the variable ϕ. They would be absent if we could use the "natural" phase ψ, and they should be minimal if W_T is close to an integer. To check this for rule $R^5_{5,9}$, we plotted in fig. 4 ΔW_T against T for such T for which the latter is true. We find exactly the predicted behaviour, giving thus a first hint that the jitter in figs. 1–3 can indeed be considered as intrinsic noise. From fig. 4 (and an analogous figure for rule $R^4_{4,8}$), we also obtain the errors in our estimates of the winding numbers,

$$w = \begin{cases} 0.6519153 \pm 0.0000007 & \text{for } R^5_{5,9} , \\ 0.6532260 \pm 0.0000009 & \text{for } R^4_{4,8} . \end{cases} \tag{9}$$

These are the values obtained for our largest lattices, but they agreed also within the errors with the values obtained from lattices with L half as big. The very small errors in eq. (9) are surprising in view of the visible jitter in figs. 2, 3. They suggest that underlying the data shown in these figures there is a much more regular process.

Let us now discuss briefly the behaviour when the initial configuration is such that again all spins are independent, but prob$(s_i = 1) = m_0 \neq 0.5$ (a more complete discussion is found in ref. [5]).

For m_0 very close to 0 or 1, all three rules lead to stationary states with small m_∞. In these states, one has essentially a dilute "gas" of stable or periodic local subconfigurations [5] similar to the various "blocks", "loaves", "beehives",

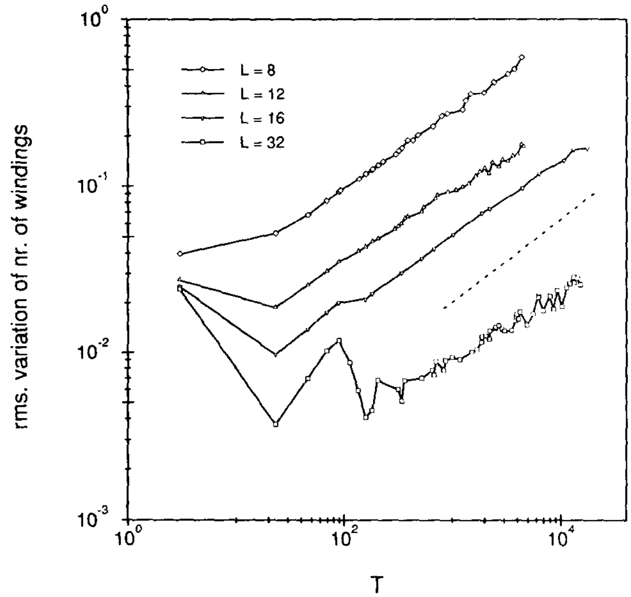

Fig. 4. log–log plot of the variation ΔW_T of the number of windings during T iterations for rule $R^5_{5,9}$ and for lattice sizes 8^5 to 32^5. For large T, the fluctuations in these curves are mostly statistical, while they are systematic for small T. The dashed line indicates a scaling law $\Delta W_T \propto \sqrt{T}$ as expected from statistical fluctuations.

"blinkers" etc. in the Game of Life [7]. For $R^4_{3,8}$ this seems to happen for $m_0 < m_- \approx 0.097$ and for $m_0 > m_+ \approx 0.992$. The closeness of these values to 0 respectively 1 might however suggest that the asymptotic thresholds (for $L \to \infty$) are indeed 0 and 1, analogous to the case of bootstrap percolation [9]. For $R^4_{4,8}$, a similar dilute state is reached for $m_0 < m_- \approx 0.291$ and for $m_0 > m_+ \approx 0.965$, while for $R^5_{5,9}$ we found $m_- \approx 0.33$ and $m_+ \approx 0.90$. If $m_0 \lesssim m_+$ or $m_0 \gtrsim m_-$, m_t first drops sharply to a value close to zero and increases after this very slowly (see fig. 5).

For rule $R^4_{3,8}$, the period-3 state is only obtained for $0.434 \lesssim m_0 \lesssim 0.524$, while a different strictly periodic state with short period (e.g. period 2 or 6, depending on the precise initial configuration) is reached otherwise. This state is characterized by localized oscillators embedded in an otherwise stationary sea [5]. Its global amplitude of oscillation seems to tend to zero for $L \to \infty$, with $m_\infty \approx 0.791$ independently of m_0. In the following we shall refer to this state as "state 4", while the three phases of the periodic pattern are called states 1, 2 and 3, and the dilute state with $m \approx 0$ is called state 0.

For the two rules with quasiperiodic behaviour, the fixed points in the centers of the circles required by topology are essentially the analogs of state 4.

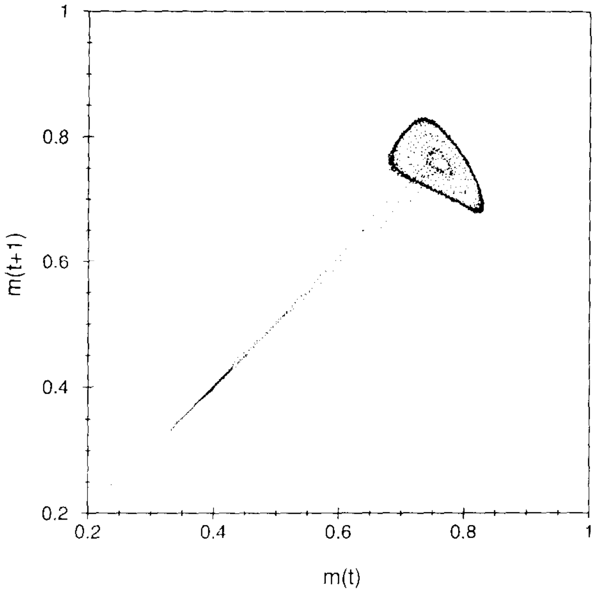

Fig. 5. m_{t+1} against m_t for rule $R^4_{4,8}$ and $L = 32$. Initial concentration was $m_0 = 0.291$, and the first 5000 iterates are plotted. Notice that the center of the circle is first approached, before the trajectory settles on the circle itself. For $m_0 = 0.305$, the center of the circle was hit much more closely.

But in contrast to rule $R^4_{3,8}$, they are unstable and go over to the quasiperiodic state (see fig. 5). Quasiperiodic behaviour was thus observed in essentially the entire intervals $m_- < m_0 < m_+$. The phase of the state after the initial transient period depends of course on m_0,

$$\lim_{t\to\infty} (\psi_t - 2\pi t w) = F(m_0) . \tag{10}$$

The function F was found to be very irregular for $R^5_{5,9}$, indicating that the phase depends quite strongly on the initial configuration (this conforms with the rich zoo of transients observed in ref. [5], depending on the detailed initial configuration). But apart from this strong phase dependence, the same state (in a statistical sense) seems to be reached in a very large range of m_0. The latter also holds for $R^4_{4,8}$. But on small lattices ($L \leqslant 16$) and for $m_0 \approx m_-$, rule $R^4_{4,8}$ leads to a variety of different states. They depend on the precise value of m_0, and actually might be very long transients with lifetimes >5000.

The behaviour of rule $R^4_{3,8}$ on small lattices will be discussed in detail in section 5.

3. Why collective behaviour is surprising

(a) Let us first recall the arguments of ref. [2] against periodic behaviour with period 3 in noisy systems. Assume we are at time t in phase 1, and we consider only the state at times $t+3$, $t+6, \ldots$. Due to spontaneous fluctuations, there will be a nonzero chance that in some region ("bubble") a phase $\neq 1$ will appear, similar to nucleation in an equilibrium situation. Phase 1 can only be stable, i.e. strict periodicity can only be maintained throughout the whole system, if such a bubble is erased again. Whether this happens depends on the velocity of the boundary (Bloch wall) between phase 1 and the new "wrong" phase. Generically, in an isotropic system this velocity will only depend on the curvature and on higher derivatives of the boundary,

$$v = v_0 + v_1/R + \cdots + \text{noise}\,, \tag{11}$$

where R is the radius of curvature and the dots stand for higher derivatives. We can assume that the sign of v_1 is such that small bubbles of wrong phase are eliminated, since otherwise even no metastable oscillations would be possible. For large enough R, the sign of v will be that of v_0. Assume now that the bubble has phase 2, and that v_0 is such that phase 1 loses against phase 2. In this case the oscillatory state is only *meta*stable, since a large enough bubble will always win, and in a large system a mixture of different phases will arise. So we must have the opposite sign of v_0 if we want strict stability. But then we consider just times $t+1$, $t+4$, $t+7, \ldots$. For these times, phase 2 is the "correct" one, and as we have just argued it will lose against a bubble with phase 1. Thus no stable oscillations are possible, unless $v_0 = 0$. But this is not to be expected generically.

Notice that this argument would not apply to period 2, since there $v_0 = 0$. Indeed, essential use was made of the fact that periodicity with period >2 implies broken detailed balance. The fact that breaking of detailed balance can wipe out structures expected from mean-field arguments was also observed in a different context in ref. [10].

For anisotropic systems these arguments need not hold if the velocity depends strongly on the orientation of the boundary [2]. Assume, for example, that phase 1 wins on a boundary perpendicular to one of the axes in 2 dimensions, while phase 2 wins on a diagonal boundary. In this case, a bubble of phase 2 will first become diamond-shaped, before it will shrink and finally disappear.

(b) Let us now generalize these arguments to quasiperiodicity. In this case, a spontaneous fluctuation will not lead to a homogeneous bubble of "wrong" phase and with sharp boundaries, but rather to a region with a small phase deviation $\delta\psi(\boldsymbol{x})$ varying smoothly with $\boldsymbol{x}$ (we use here a continuous space variable $\boldsymbol{x} \in \mathbb{R}^d$). Instead of eq. (11), we have now a dependence of the winding number w on the gradient of $\psi(\boldsymbol{x})$,

$$\dot{\psi}(\boldsymbol{x}, t) = 2\pi w_0 + \frac{w_1}{2\pi} [\nabla\psi(\boldsymbol{x}, t)]^2 + w_2 \nabla^2 \psi(\boldsymbol{x}, t) + \cdots + \text{noise} , \tag{12}$$

where w_0 is the winding number for constant phase discussed in the previous section. Just like the curvature term in eq. (11), the term proportional to $\nabla^2\psi$ can be neglected for sufficiently smooth fluctuations except when the dominant term vanishes, i.e. when $w_1 = 0$.

Assume now that $\delta\psi(\boldsymbol{x}) > 0$ (see fig. 6). Then the bubble will spread with velocity $\boldsymbol{v} = w_1 \nabla\psi$, and $\delta\psi(\boldsymbol{x})$ will grow if $w_1 > 0$. Similarly, a fluctuation with $\delta\psi(\boldsymbol{x}) < 0$ will grow if $w_1 < 0$. Stability will be reached only if neither happens, i.e. if $w_1 = 0$. But this is not to be expected generically. We should point out that eq. (12) is formally just the Kardar–Parisi–Zhang equation [11] for the growth of a surface. Simulations and heuristic arguments [12] show that it leads indeed to unbounded fluctuations of ψ on an infinite lattice.

Notice that here only the behaviour of the winding number with respect to long wavelength perturbations is needed. This should depend less on any lattice anisotropy than the velocity of the sharp kinks discussed in the context of periodicity.

In order to estimate w_1 for rule $R^5_{5,9}$, we performed simulations on a bar-shaped lattice of size $L^4 \times L'$, with $L' \gg L$. The initial configuration was

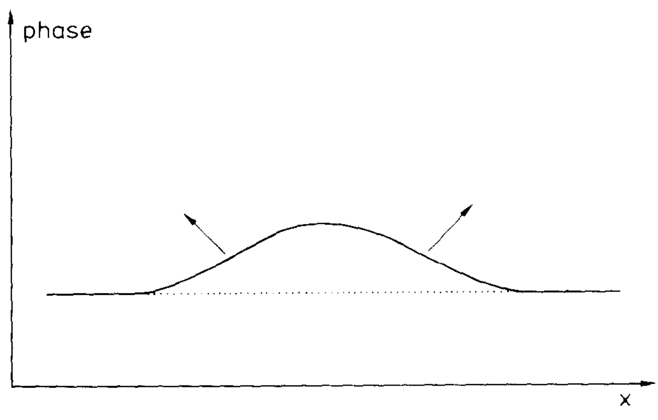

Fig. 6. Schematic drawing of a positive phase fluctuation in a one-dimensional system. If the winding number grows with the gradient of the phase, then the boundaries of the fluctuation will move apart and the fluctuation will grow. Similarly, a negative fluctuation would grow if the winding number would decrease with $|\nabla\psi|$.

carefully constructed such that the phase increased in small steps along the long axis with $\psi(L') - \psi(0) = 2\pi$, implying a gradient $\mathrm{d}\psi/\mathrm{d}x = 2\pi/L'$. Due to the periodic boundary conditions, this gradient is maintained during the evolution, provided $\psi(x)$ stays everywhere well defined. We checked that this was indeed the case for $L' = 112$ and $L' = 198$, and we obtained in this way w_1.

Unfortunately, these results are not very precise since these simulations could not be done efficiently on the Connection Machine. Yet we obtained a nonzero value $w_1 = -0.4 \pm 0.1$. Thus the quasiperiodic motion should be unstable against fluctuations with $\delta\psi > 0$. Notice however that w_1 is small, indicating that this process should be very slow. On the other hand, these simulations showed that a phase gradient is maintained during the evolution (for 2×10^4 time steps at least), stressing again the robustness of the quasiperiodicity.

Before leaving this section, we should point out that a term proportional to $(\nabla\psi)^2$ should also be generically present in models describing extended oscillatory phenomena with non-constant amplitude. Such phenomena are usually modeled by the complex Ginzburg–Landau equation (see e.g. refs. [13–15]) which does not contain such a term. Including it should change the behaviour of the phase transition observed in this model. Thus we conjecture that the behaviour at the onset of chaos found in refs. [14, 15] is not the generic one.

4. Statistical properties of the intrinsic "noise"

We studied quite thoroughly the statistical properties of the jitter seen in figs. 1–3. On the one hand, the arguments of the last section depend on the assumption that it is equivalent to true noise without any hidden determinism. On the other hand, a priori it might appear strange that a deterministic and discrete system could show stochastic behaviour.

Indeed, the latter is not really forbidden. It is true that the above CA on finite lattices cannot behave stochastically. But on infinite lattices, one can find similar stochastic behaviour as in deterministic chaotic systems [16, 17]. For the times feasible in our simulations, this might lead to effectively chaotic motion even on the finite lattices studied in this paper. The essential question then is whether this is a low dimensional attractor or not. In the latter case its randomness would resemble that of a pseudorandom number generator.

The most straightforward test for randomness in time sequences consists in looking at power spectra. For rule $R^4_{3,8}$, the spectrum is shown in fig. 7. In order to suppress the trivial period-3 component, we show in this figure indeed the sum of the 3 spectra of the sequences $\{m_{3t}\}$, $\{m_{3t+1}\}$, and $\{m_{3t+2}\}$. We see no peak indicative of any additional discrete frequency, though the noise is not

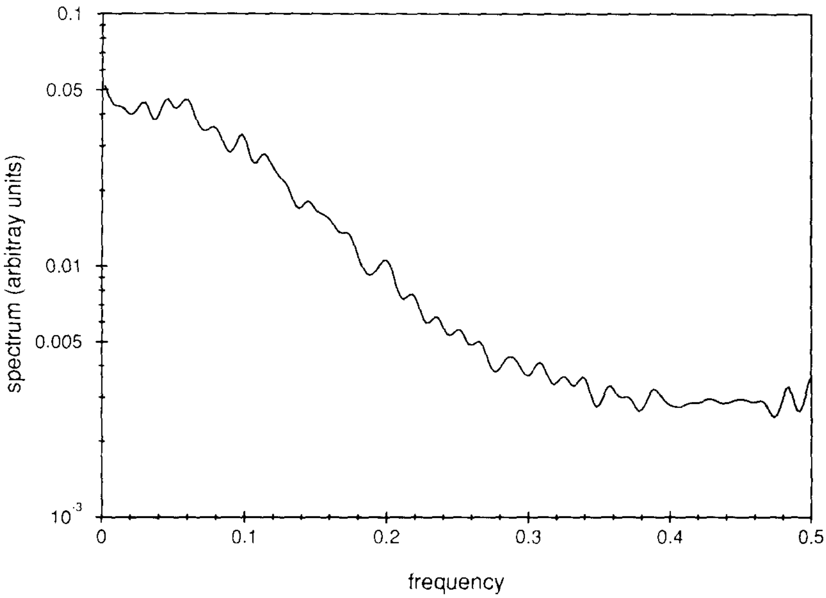

Fig. 7. Sum of the power spectra of the three subsequences $\{m_{3t+k}\}$, $k = 0, 1, 2$, generated by $R^1_{3,8}$ with $L = 64$. Data are from 10^5 iterations, the first 1000 of which were discarded.

at all white. The enhancement of the low-frequency region is also seen directly, in rather strong correlations.

It is well known that looking at power spectra can be misleading if the system is chaotic. In order to test for hidden deterministic behaviour in the noisy quasiperiodic cases, we used a method due to Badii and Politi [18]. We first represented the state at time t by a D-dimensional delay vector $\boldsymbol{m}_t = (m_t, m_{t-1}, \ldots, m_{t-D+1})$. Then, we measured the average distances between kth nearest neighbours among N such vectors. Let us denote by $r_t^{(k)}$ the (Euclidean) distance of the kth neighbour from $\boldsymbol{m}_t$. On large scales, the fuzziness of the loops in figs. 2 and 3 can be neglected and these vectors are lined up on curves. Their distances should then scale with N as

$$\langle r^{(k)} \rangle \sim 1/N \qquad (\text{small } N) . \tag{13}$$

A different scaling law should hold on small distances where the fuzziness of the lines in figs. 2 and 3 dominates. If that fuzziness is due to random noise, then the vectors should fill up a D-dimensional volume, in which case

$$\langle r^{(k)} \rangle \sim N^{-1/D} \qquad (\text{large } N) \tag{14}$$

on small scales. If, in contrast, the jitter is due to low dimensional chaos with dimension $D' < D$, we would have $\langle r^{(k)} \rangle \sim N^{-1/D'}$ for large N respectively small r.

A certain problem in verifying eq. (14) is that we have to go to very small distances in order to avoid the trivial scaling of eq. (13). This implies that we need very large N, not too large lattice sizes, and not too large embedding dimensions D. Results for $R^5_{5,9}$ with $L = 16$ and $k = 3$ are shown in fig. 8 for various D. The high statistics there were possible due to the fast neighbour search described in ref. [19]. We see nice agreement with eqs. (13), (14) which suggests that the jitter behaves indeed like stochastic noise. Similar results were obtained for $R^4_{4,8}$ and $R^4_{3,8}$.

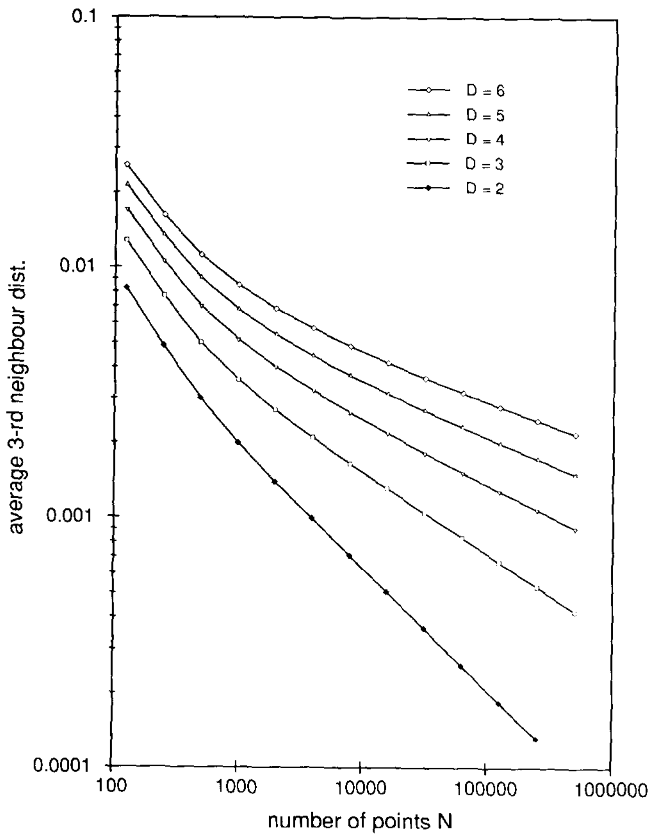

Fig. 8. Average third nearest neighbour distances among delay-space reconstructed states of $R^5_{5,9}$, plotted versus the number N of points. Embedding dimensions range from $D = 2$ to $D = 6$. Data are from 500 000 iterations on a 16^5 lattice, cut into pieces of length N after discarding the first 1000 iterations.

Next, we studied the dependence of the noise amplitude on the lattice size L, by comparing the cloud sizes (for $R^4_{3,8}$) as well as the third nearest neighbour distances (for $R^4_{4,8}$ and $R^5_{5,9}$) for different L. Results for the 4D rules embedded in $D=2$ are shown in figs. 9 and 10.

For rule $R^4_{3,8}$ we found that the shapes of the clouds are independent of L, and that their diameters (radii of gyration) scale as

$$R \sim L^{-\alpha}, \qquad \alpha = 2.003 \pm 0.010. \tag{15}$$

This can be easily understood if we assume that distant parts of the lattice are only weakly coupled. In this case, m is an average of $n \propto L^d$ essentially independent contributions. According to the central limit theorem, we should then expect the fluctuations δm of m to scale as $L^{d/2}$. This argument is further evidence that the jitter is essentially stochastic noise. It should of course not be taken too literally, as it could not explain why different parts of the lattice remain coherent.

For $R^4_{4,8}$, the third neighbour distances scale as

$$\langle r^{(3)} \rangle \sim L^{-\beta}, \qquad \beta = 0.976 \pm 0.006. \tag{16}$$

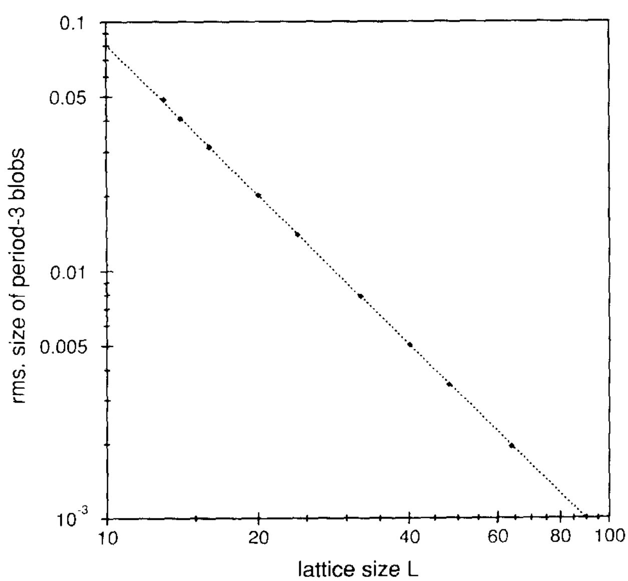

Fig. 9. Average root mean squared radii of the three clouds for rule $R^4_{3,8}$, plotted on a log–log plot versus lattice size L.

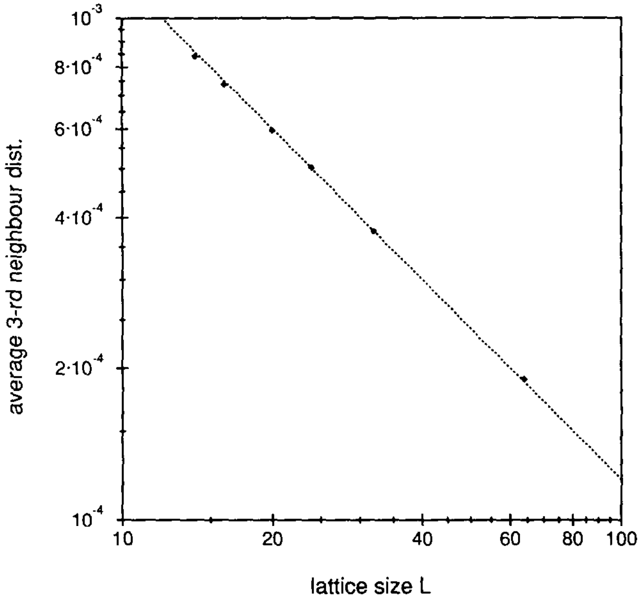

Fig. 10. Average third neighbour distances among 32 000 points for rule $R^4_{4,8}$, plotted on a log–log plot versus lattice size L (dots; the error bars are smaller than the dot sizes). The dotted line corresponds to eq. (14) with $\beta = 1$.

An argument analogous to the previous one gives here that the width of the curves in fig. 2 should scale $\sim L^{-d/2}$, giving $\beta = d/4 = 1$. This time the agreement is not perfect, suggesting that the loose coupling of different parts of the lattice might lead to small phase differences between them. For very large lattices, this would suggest that coherence could be lost completely.

We do not show the analogous plot for rule $R^5_{5,9}$, as it would look very much like fig. 10. Instead, we show in fig. 11 immediately the local slopes in this plot, which should give $-\beta$ in the limit $L \to \infty$. The above argument would give $\beta = 1.25$. For small lattices, we see that the observed value is slightly smaller (as for rule $R^4_{4,8}$), but it becomes larger for large lattices. If this would hold also for $L \to \infty$, it would suggest that distant parts of the lattice are strongly coupled, and that the coherent behaviour is more stable in 5 dimensions than in 4 dimensions. Though this agrees with our intuition that fluctuations are less important in higher dimensions we should point out that the effect is statistically barely significant.

Let us finally come back to the fluctuations of the winding numbers shown in fig. 4. According to the above naive arguments, they should scale like $\Delta W_T \propto L^{-5/2}$ for large T. This is indeed seen in fig. 4.

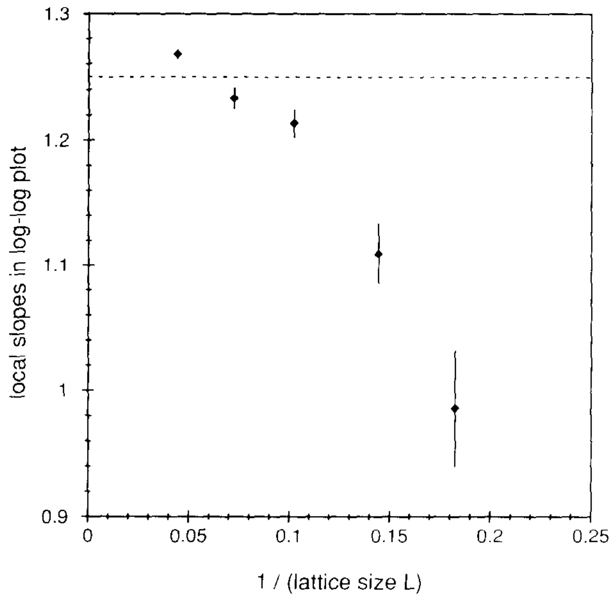

Fig. 11. Local slopes in a log–log plot like fig. 9, but for rule $R^5_{5,9}$. On the abscissa, we show $1/L$. The exponent β is obtained by extrapolating these slopes to $L \to \infty$. The dashed line is the prediction $\beta = 5/4$.

5. Metastability

As we have already noted, the collective behaviour seems to be metastable in all three rules, if we consider small lattices or systems corrupted by external noise. In this section, we shall study in detail only rule $R^4_{3,8}$.

Let us first consider the case without external noise. A typical decay of a metastable state on a lattice with $L = 11$ is shown in fig. 12. From this we see a rather sharp transition, giving a rather well defined lifetime. The exact lifetime will of course depend on our exact criterion for when the state has definitely decayed, but this criterion will not affect the constant T_0 in the lifetime distribution,

$$P(t) \equiv \text{prob}(T > t) \sim e^{-t/T_0}, \qquad t \to \infty . \tag{17}$$

This distribution is shown in fig. 13 for $L = 10$, where indeed an exponential decay as postulated in eq. (17) is seen, except for very small t.

The dependence of T_0 on L is shown in fig. 14. For $L \geqslant 9$, we see a perfect straight line when plotting T_0 versus L^2, suggesting

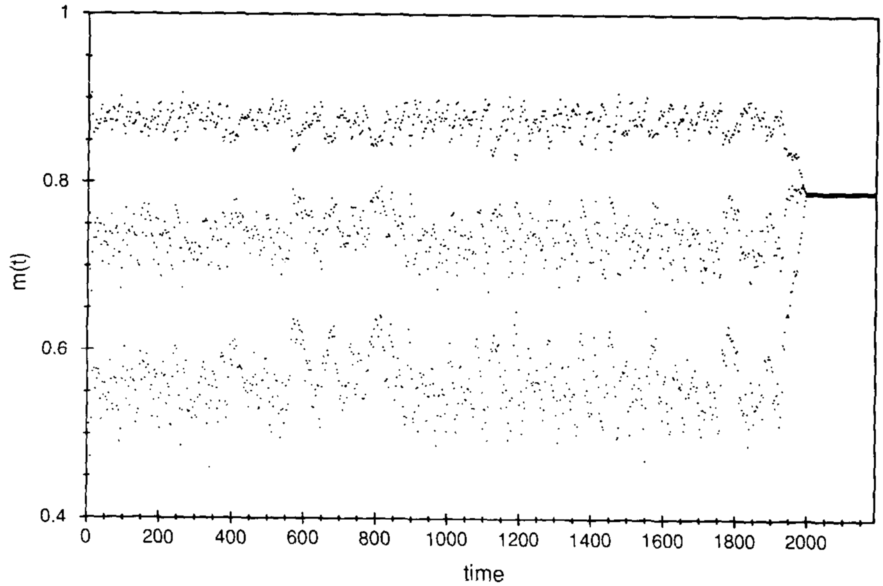

Fig. 12. m_t versus t for $R^4_{3,8}$ and $L = 11$. Notice the rather sharp transition from periodic to fixed point behaviour typical for the decay of a metastable state.

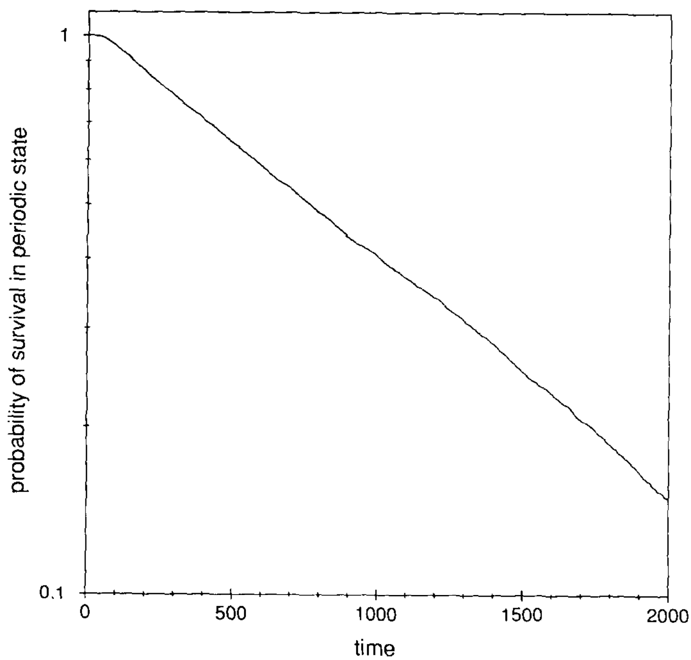

Fig. 13. Survival probability $P(t)$ versus t for rule $R^4_{3,8}$ with $L = 10$, based on 5375 independent runs, on a logarithmic plot.

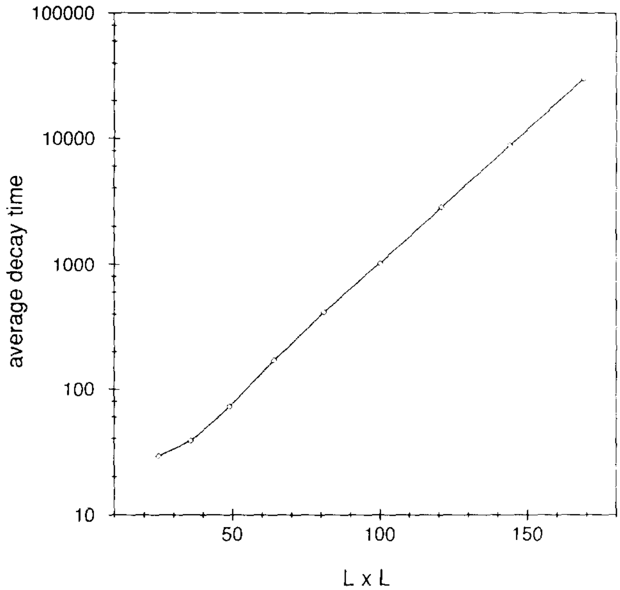

Fig. 14. Decadic logarithm of the inverse decay rate T_0, defined in eq. (18), versus the square of the lattice size L. The number of lattices simulated varied between 8000 (for $L = 8$) and 708 (for $L = 13$). The error bars are smaller than the dots. The rule is $R^4_{3,8}$.

$$\log T_0 \propto L^2 \qquad \text{for } L \to \infty . \tag{18}$$

This would mean that the collective behaviour is stable on infinite lattices, in contrast to what is suggested by the arguments of section 2.

According to these arguments, we would expect that the decay of metastable states occurs via the formation of bubbles of radius $R > R_0$, provided of course $L > R_0$. If L is less than the critical radius R_0, then the decay occurs via a coherent flip of the entire lattice. If this flip occurs due to stochastic noise – the main conclusion of the last section was that the observed jitter is indeed like noise – and if the noise level scales as $1/L^2$, then we obtain immediately eq. (18). From this we can conclude that $R_0 \gg L$.

For a more precise estimate, we need an assumption on what happens for $L \approx R_0$. A simple ansatz is

$$\log T_0 \propto \frac{L^2 R_0^2}{L^2 + R_0^2} . \tag{19}$$

Using this ansatz, we find that within 95% confidence level $R_0 > 40$ and

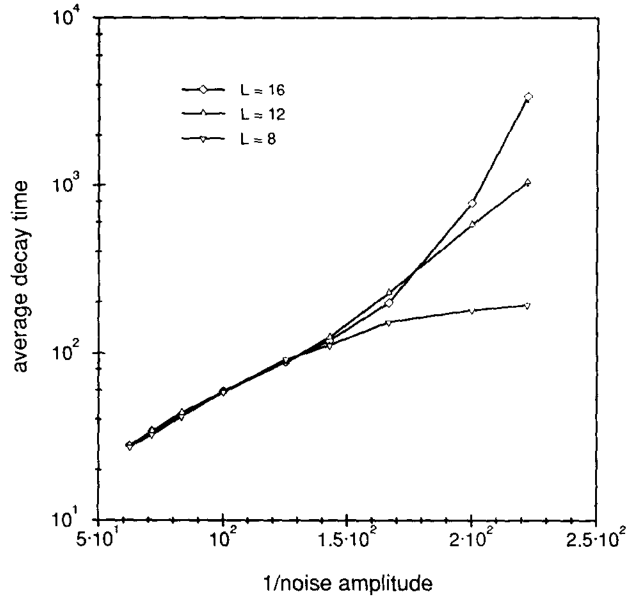

Fig. 15. Logarithms of average lifetimes versus inverse noise amplitude p^{-1}, for lattice sizes $L = 8$, 12, 16. The rule is $R^4_{3,8}$.

$\lim_{L\to\infty} T_0 > 10^{41}$, showing that the periodic behaviour is stable for all practical purposes. Our best estimate from this fit is $R_0 = \infty$.

If our picture is correct that the observed jitter is equivalent to intrinsic stochastic noise, then addition of external noise should just add to the total noise level and should thus suppress the lifetime. This was indeed seen. We added external noise by modifying eq. (1): we used eq. (1) only with probability $1 - p$, while we did the opposite move, $s_i^{t+1} = 1 - f(\Sigma_{\langle i,j\rangle} s_j^t)$, with probability p. Results for the average lifetime are shown in fig. 15. We now do not see simple scaling with the inverse noise amplitude, and also the lifetime distributions were far from exponential. But we do observe the expected monotonic decrease of the lifetime with the noise amplitude p.

6. Phase boundaries

In this section, we finally report on simulations of rule $R^4_{3,8}$ where we first let the system run into the period-3 state, and then iterated only part of the lattice one step further. In this way, we produced artificially boundaries between two different phases. According to the arguments of ref. [2] recalled in section 3, stability of the oscillation is to be expected only if these boundaries do not move, or if it depends on the orientation of the boundary which of the two phases wins.

In these simulations, we verified results found also in ref. [5]: if the boundary is perpendicular to one of the axes, neither phase wins. Instead, it is state 4 which is formed along the boundary and then spreads with equal speed in both directions. More precisely, we measured a velocity of spreading $v = 0.013 \pm 0.003$ lattice units per time unit. We should add that the spreading was rather regular, without the large fluctuations which would be expected if state 4 were only marginally more stable than the periodic states. The same effect (with similar speed of spreading) was observed when the boundary was oriented perpendicular to the vector $(1, 1, 0, 0)$.

Though we have not looked at all different orientations, we thus find it unlikely that there exist directions for which state 4 will not spread.

We might add that we have also printed out cuts through the lattice. On these, we did not see any conspicuous anisotropy. Nor did we see any patterns or any regularities in any of these cuts.

7. Conclusions

In the last section, we have seen that the arguments of ref. [2] should apply to rule $R^4_{3,8}$, though with an important modification. It is not bubbles with wrong phases which should ruin the coherence, but bubbles with the stationary state 4. The result should, however, be the same: the periodic state should only be metastable.

This argument gives us no estimate for the lifetime of this metastable state, since we do not know how the velocity of spreading depends on the curvature of the bubble's boundary. Yet it seems hard to believe that the radius beyond which state 4 wins should be as large as estimated in section 5. The lifetimes on small lattices presented in that section clearly are most easily explained if the lifetime diverges as $L \to \infty$. We consider this as a puzzle which cannot easily be solved by numerical simulations.

The other conclusion of this paper is that the jitter seen in the simulations is statistically indistinguishable from stochastic noise. Its lattice-size dependence shows scaling relations which can be understood by very naive arguments. But these arguments seem to be mutually inconsistent. In particular, except for very small deviations which might not be statistically significant, these scaling relations would follow if large lattices effectively decouple. If this were true, coherence would be lost very soon.

Regardless of these problems, we found that the periodic and quasiperiodic behaviour found by Chaté and Manneville extends to much longer times and to much larger lattices than studied by these authors. Whether they represent true asymptotic behaviour or just metastable states is somewhat irrelevant in view of their very long lifetimes. We conclude thus that this behaviour is very interesting, and it would be nice to observe it also in lower dimensions and in realistic physical systems.

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