J. Phys. B: At. Mol. Phys. **15** (1982) L75–L80. Printed in Great Britain

LETTER TO THE EDITOR

On the spacing of the quasi-Landau resonances

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Received 29 October 1981

Abstract. Simple analytical expressions for the spacing between the quasi-Landau resonances, covering the whole energy range are derived. The expressions are obtained in a first-order WKB approximation considering the $m = 0$ spinless electron to move in the $z = 0$ plane. In addition, we show a crossing over of the spacing at negative energies and that the spacing is not very sensitive to the presence of a centrifugal barrier.

After the experiments of Garton and Tomkins (1969), much work was concentrated on the study of the spacing between the quasi-Landau resonances at zero energy. This was motivated by their observation of the absorption spectrum of Ba in a magnetic field $B = 24$ kG in which broad resoances spaced by approximately $1.5\ \hbar\omega$ were found near $E = 0$ ($\omega = eB/Mc$ is the cyclotron frequency where M is the electron mass). This striking feature of the spectrum was explained through a semiclassical argument by O'Connell (1974). WKB explanations of the same fact were provided by Edmonds (1970), Starace (1973) and by Rau (1979). An account of the current status of this area may be found in the work of Gay (1980).

The main problem in studying the motion of a spinless electron moving in the combined Coulomb and magnetic fields is that the resulting Schrödinger equation for the system is not separable. Indeed, if one follows Landau and Lifshits (1977) and writes the wavefunction of the electron in cylindrical coordinates as (Starace 1973)

$$\psi(\rho, \phi, z) = \rho^{-1/2} f(\rho, z) \exp(im\phi) \tag{1}$$

then it is easy to see that the Schrödinger equation of the problem at hand reduces to

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{\partial^2 f}{\partial z^2} + \frac{2M}{\hbar^2}(E - V(\rho, z))f = 0 \tag{2}$$

where

$$V(\rho, z) = \frac{\hbar^2}{2M}\frac{T}{\rho^2} - \frac{e^2}{(\rho^2 + z^2)^{1/2}} + \frac{1}{8}M\omega^2\rho^2 \tag{3}$$

and where $T = m^2 - \frac{1}{4}$, m being the magnetic quantum number of the electron. In equation (2) we have used the same symbol E to represent the energy shifted by $-\hbar\omega m/2$. The complicated two-dimensional problem defined in equation (2) can be reduced to a tractable one-dimensional problem by studying the motion of the electron

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0022-3700/82/030075+06$02.00

in the $z = 0$ plane. This approximation, previously used in the works of Starace (1973) and Rau (1979), successfully predicts the experimental positions of the resonances as reported by Economou *et al* (1978) and, more recently, by Fonck *et al* (1980). In the present work we also assume this approximation to be valid so that the two-dimensional potential of equation (2) reduces to the one-dimensional potential

$$V(\rho) = \frac{\hbar^2}{2M}\frac{T}{\rho^2} - \frac{e^2}{\rho} + \frac{1}{8}M\omega^2\rho^2 \tag{4}$$

where $T = m^2 - \frac{1}{4}$ was replaced by $T = m^2$ because of the criterion for the applicability of the WKB approximation (Langer 1937). It is interesting, however, to note that different expressions for T were reported to reproduce experimental results correctly: using $T = (m+\frac{1}{2})^2$ instead of $T = m^2 - \frac{1}{4}$ in equation (4) Economou *et al* (1978) obtained good agreement for the $m = 1$ and $m = 2$ sublevels of rubidium at 0 and 49 kG. More recently Fonck *et al* (1980) using $T = 0$ found good agreement for the $m = 0$ and $m = -1$ sublevels in barium and strontium at 25, 40 and 47 kG. Fonck *et al* preferred to use $T = m^2 = 0$ in their model of the electronic motion and attributed the good agreement obtained by Economou *et al* to an accidental compensation occurring for the spin-$\frac{1}{2}$ Rb I atom. Anyway we observe that since the T/ρ^2 term dominates at the origin the characteristics of the one-dimensional motion will depend very much on whether T is positive, negative or zero.

As mentioned before, the spacing of quasi-Landau resonances was previously studied in a WKB approximation by Starace (1973) and Rau (1979). Starace calculated the spacing of the resonances numerically in the region of $E \geqslant 0$, modelling the electronic motion by equation (4) with $T = 1$. Rau, on the other hand, showed that for the particular case $E = T = 0$ the WKB integral reduces to a trivial one which was then evaluated analytically. The correct $1.5\hbar\omega$ spacing at $E = 0$ was obtained by both authors in spite of the different nature of the potentials used (the presence or not of the centrifugal term).

In the present paper we assume $m = 0$ and derive analytical expressions for the spacing between the quasi-Landau resonances. These expressions cover the whole energy range and reproduce the $1.5\hbar\omega$ spacing at $E = 0$ as expected. In particular our expressions are also valid for negative energies for which interesting experimental results have been recently reported (Gay *et al* 1980, Delande and Gay 1981).

For $m = 0$ equation (4) reduces to

$$V(\rho) = -\frac{e^2}{\rho} + \frac{1}{8}M\omega^2\rho^2. \tag{5}$$

The WKB quantisation rule applied to this potential is

$$(2M)^{1/2}\int_0^{\rho_0}\left(E + \frac{e^2}{\rho} - \frac{1}{8}M\omega^2\rho^2\right)^{1/2} d\rho = (n + 1/2)\pi\hbar \tag{6}$$

where ρ_0 is the only positive root of

$$E + \frac{e^2}{\rho} - \frac{1}{8}M\omega^2\rho^2 = 0. \tag{7}$$

A simple differentiation of equation (6) with respect to n gives the spacing as

$$\frac{dE}{dn} = \frac{2\pi\hbar}{(2M)^{1/2}}\left[\int_0^{\rho_0}\left(E + \frac{e^2}{\rho} - \frac{1}{8}M\omega^2\rho^2\right)^{-1/2} d\rho\right]^{-1}. \tag{8}$$

In figure 1 a schematic view of the potential of equation (5) is given. Although physically meaningless the region $\rho < 0$ is also included. $V(\rho)$ has a relative minimum $V_c = (27M\omega^2 e^4/32)^{1/3} > 0$ at $\rho_c = -(4e^2/M\omega^2)^{1/3}$. From this figure it is clear that equation (7) can be factored in two different ways depending on the relative magnitudes of E and V_c. Besides the real positive root ρ_0, equation (7) may have: (i) for $E \leqslant V_c$, two complex conjugate roots and (ii) for $E > V_c$, two negative and different real roots. The case $E \leqslant V_c$ includes the particular value $E = 0$ studied by Rau (1979). Now, since we know the nature of the roots of equation (7) for any value of E the integrand in equation (8) can be factored accordingly. The resulting integrals are trivial elliptic integrals.

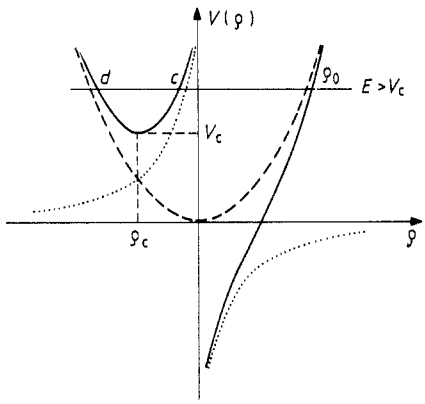

Figure 1. Coulomb plus magnetic potentials as given by equation (5) (Coulomb field: . . . ; magnetic field: - - - ; sum: ——). For $E > V_c$ the three roots of $E - V(\rho) = 0$ are real and different. For $E \leqslant V_c$ one root is real and the other two are complex conjugate.

For $E \leqslant V_c$ equation (8) can be written as

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left(\int_0^{\rho_0} \frac{\rho\, d\rho}{\{\rho(\rho_0-\rho)[(\rho-b)^2+a^2]\}^{1/2}}\right)^{-1} \tag{9}$$

where a, b and ρ_0 are obtained from the solution of equation (7), namely,

$$-\rho^3 + \frac{8E}{M\omega^2}\rho + \frac{8e^2}{M\omega^2} \equiv (\rho_0-\rho)[(\rho-b)^2+a^2] = 0. \tag{10}$$

Since the above polynomial does not contain a second-degree term it follows that $b = -\rho_0/2$. The integral in equation (9) can then be easily evaluated in terms of complete elliptic integrals. The result is

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left(\frac{\rho_0(AB)^{-1/2}}{A-B}[-2B\mathbf{K}(k) + (A+B)\mathbf{\Pi}(\alpha^2, k)]\right)^{-1} \tag{11}$$

where

$$A^2 = (9\rho_0^2 + 4a^2)/4 \qquad B^2 = (\rho_0^2 + 4a^2)/4$$
$$\alpha^2 = -(A-B)^2/4AB \qquad k^2 = [\rho_0^2 - (A-B)^2]/4AB$$

and where $\mathbf{K}(k)$ and $\mathbf{\Pi}(\alpha^2, k)$ are the complete elliptic integrals of the first and third kind, respectively (Byrd and Friedman 1975).

For $E > V_c$ equation (7) factors as

$$-\rho^3 + \frac{8E}{M\omega^2}\rho + \frac{8e^2}{M\omega^2} \equiv (\rho_0 - \rho)(\rho - c)(\rho - d) \tag{12}$$

where $d < c < 0 < \rho_0$ (see figure 1). It then follows

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left(\int_0^{\rho_0} \frac{\rho\, d\rho}{[\rho(\rho_0 - \rho)(\rho - c)(\rho - d)]^{1/2}}\right)^{-1}$$

$$= \hbar\omega \frac{\pi}{2}\left(\frac{2}{\sqrt{-d(\rho_0 - c)}}[d\mathbf{K}(k) + (\rho_0 - d)\mathbf{\Pi}(\rho_0/d, k)]\right)^{-1} \tag{13}$$

where $k^2 = [(c-d)\rho_0]/[-d(\rho_0 - c)]$.

Equations (11) and (13) are the main results of this letter. The elliptic integrals in these expressions can be very easily and accurately calculated with the algorithms of Carlson (1979).

It is not difficult to see that when $E = V_c$ the complex conjugate roots have zero imaginary part ($a = 0$ in equation (10)) and the integral of equation (9) reduces itself to a pseudo-elliptic one. In this case the spacing can be written in terms of elementary functions and simplified to

$$\frac{dE}{dn} = \hbar\omega(3 + 3^{1/2})/4 \simeq 1.183\hbar\omega. \tag{14}$$

The values $E = V_c$ and $E = 0$ are the only ones for which equation (9) can be reduced to pseudo-elliptic integrals.

Figure 2(a) shows the energy spacings of the quasi-Landau resonances as obtained from equations (11) and (13) above. In this figure the spacings were calculated for the same field values as Starace (1973). Note, however, that Starace considered the case $T = 1$ while we are presently assuming $T = 0$. As is easy to see from figure 2 the spacing of $1.5\hbar\omega$ at $E = 0$, first obtained for $T = 0$ by Edmonds (1970) and O'Connell (1974), is reproduced here. By comparing figure 2(a) with figure 1 of Starace it is easy to see that although the $T = 1$ and $T = 0$ models are very different (the presence or not of the centrifugal term) they predict roughly similar spacings for $E > 0$. We also calculated the spacing for negative energy values. These are shown in figure 2(b). It is interesting to observe the crossing over of the constant B lines and that the energy spacing increases as E decreases. These conclusions agree with O'Connell's semiclassical treatment (see his equation (9)).

In figure 3 we give the spacing of the quasi-Landau resonances, for arbitrarily fixed energy values, as a function of the magnetic field. In this figure we see the asymmetry of the spacing with respect to the $E = 0$ line.

In the present letter we derived analytic expressions for the spacing of the quasi-Landau resonances within the first-order WKB approximation. The expressions are valid for a $m = 0$ spinless electron moving in the $z = 0$ plane, as usual. The expressions extend the work of Rau out of the $E = 0$ case. By studying the behaviour of the spacing as a function of the energy we found a crossing over at negative energies for any pair of B fields. Finally we observe that similar expressions to ours have been derived by Akimoto and Hasegawa (1967) and that their results are incorrect. From their equations (4.2), (A.16), (A.17) and (3.2) one finds $dE/dn = \hbar\omega/2g(t)$. When $E = 0$,

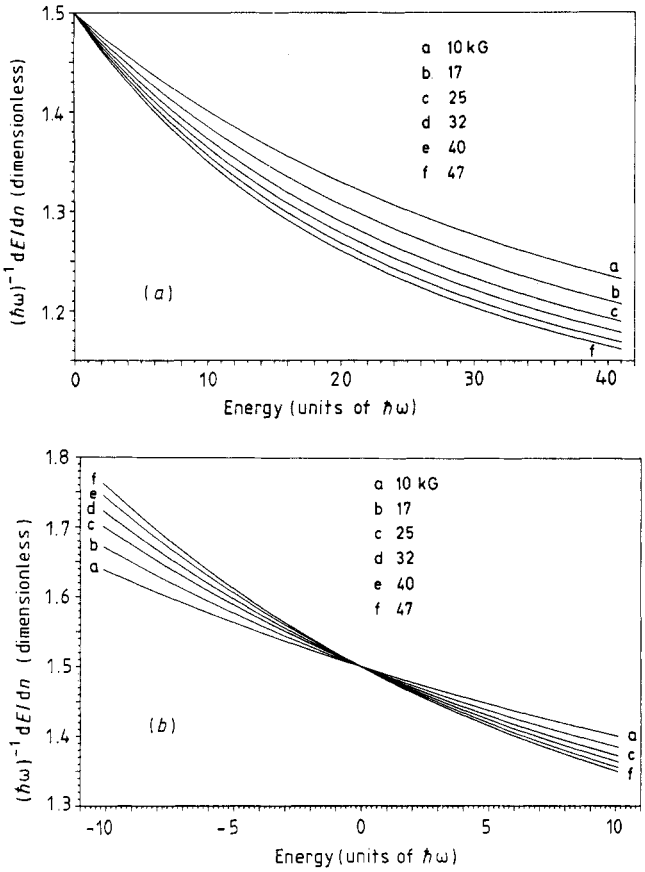

Figure 2. Plot of the energy spacing dE/dn against energy E for $B = 10, 17, 25, 32, 40$ and 47 kG. Both dE/dn and E are in units of the cyclotron energy $\hbar\omega = e\hbar B/Mc$. (*a*) Behaviour above the ionisation limit ($E = 0$); (*b*) crossing over at $E = 0$. Note the asymmetry at negative energies.

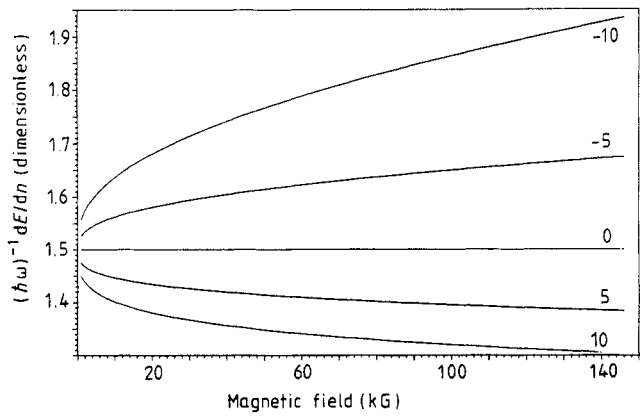

Figure 3. Plot of the spacing of the quasi-Landau resonances against magnetic field B for $E = -10, -5, 0, 5$ and 10 in units of $\hbar\omega$.

according to their equations (A.3) and (A.8), $t = 1$ and from (A.18) follows $g(1.0) \simeq 0.5994$ giving $dE/dn \simeq 0.8342\hbar\omega$ which is incorrect.

JACG acknowledges support from UFSC-Florianópolis/Brasil and from DAAD/Germany. He also thanks Professor E Werner for discussions. RFO'C acknowledges support from the Department of Energy, Division of Material Sciences, Contract DE-AS05-79ER10459. He would also like to thank the Max-Planck Institute of Quantum Optics for hospitality.

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