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LETTER TO THE EDITOR

Effect of the magnetic quantum number on the spacing of quasi-Landau resonances

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Abstract. Simple analytical expressions for the spacing between the quasi-Landau resonances are derived in a first-order WKB approximation. The expressions are valid for any value of the magnetic quantum number and cover the whole energy range. In addition, several aspects of the effect of the magnetic quantum number on the spacing of the quasi-Landau resonances are discussed.

In a recent paper (Gallas and O'Connell 1982, hereafter referred to as I) analytical expressions for the spacing between the so called quasi-Landau resonances were derived. The spacing was obtained through a first-order WKB approximation considering the spinless electron as constrained to move in the $z = 0$ plane. (For details see I and the review article of Garstang 1977; the present status of the field is discussed by Gay 1980.) In I the magnetic quantum number of the electron was assumed to be $m = 0$, which means that the centrifugal barrier present in the WKB model is neglected. This previous work is now extended by taking the centrifugal barrier into consideration and quantitatively studying the influence of the magnetic quantum number on the spacing. This extension is motivated in part by the different *Ansätze* used by experimental groups in interpreting their observations (Economou *et al* 1978, Fonck *et al* 1980, Gay *et al* 1980).

As discussed in I, the electron of mass M 'sees' the potential

$$V(\rho) = \frac{\hbar^2}{2M}\frac{T}{\rho^2} - \frac{e^2}{\rho} + \frac{1}{8}M\omega^2\rho^2 \tag{1}$$

where $\omega = eB/Mc$, B being the magnetic field; T is a known function of the magnetic quantum number m. The Langer-transformed Schrödinger equation gives $T = m^2$. However, the expressions for the spacing to be derived here will not depend on any particular functional relation between T and m.

Following I, we plot in figure 1 the potential of equation (1), emphasising the points of interest for us here. For this potential the quantisation rule is given by

$$\frac{M\omega}{2}\int_{\rho_1}^{\rho_2}(-\rho^4 + p\rho^2 + q\rho - r)^{1/2}\frac{d\rho}{\rho} = (n + \tfrac{1}{2})\pi\hbar \tag{2}$$

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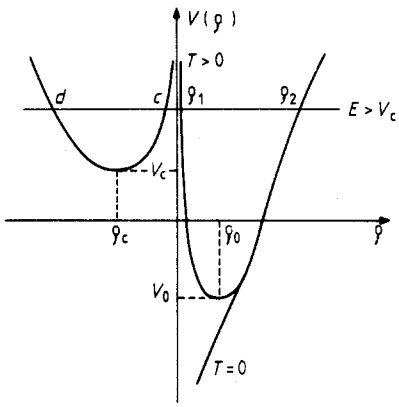

Figure 1. Coulomb plus magnetic potentials as given by equation (1).

where $p = 8E/M\omega^2$, $q = 8e^2/M\omega^2$ and $r = 4\hbar^2 T/(M\omega)^2$, E being the energy of the electron including the paramagnetic shift $-m\hbar\omega/2$. Under the usual assumption that E is a differentiable function of n, the spacing can easily be obtained from equation (2):

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left(\int_{\rho_1}^{\rho_2} (-\rho^4 + p\rho^2 + q\rho - r)^{-1/2} \rho \, d\rho \right)^{-1}. \tag{3}$$

In what follows we assume $T \geqslant 0$ since this interval contains all functional relations $T = T(m)$ hitherto used to fit the experimental data. The case $T = 0$ was considered in I.

From figure 1 it is easy to see that for any energy value such that $E > V_0$ the polynomial in parentheses in equation (3) will always have two real and positive roots, say $\rho_1 < \rho_2$. The nature of the other two roots depends on the relation between E and V_c: for $E > V_c$ they are real, negative and different, say $d < c < 0$, and for $E \leqslant V_c$ they are complex conjugate numbers. All roots are found by solving the equation

$$-\rho^4 + p\rho^2 + q\rho - r \equiv (\rho - \rho_1)(\rho_2 - \rho)[(\rho - b^2) + a^2] = 0 \tag{4}$$

when $E \leqslant V_c$ (b and a are, respectively, the real and imaginary parts of the complex conjugate roots), or

$$-\rho^4 + p\rho^2 + q\rho - r \equiv (\rho - \rho_1)(\rho_2 - \rho)(\rho - c)(\rho - d) = 0 \tag{5}$$

when $E > V_c$. Using these definitions the integral in equation (3) can be shown to be

(i) for $V_0 < E \leqslant V_c$:

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left(\frac{(AB)^{-1/2}}{A - B}[2(\rho_1 A - \rho_2 B)\mathbf{K}(k) + (\rho_2 - \rho_1)(A + B)\mathbf{\Pi}(\alpha^2, k)] \right)^{-1} \tag{6}$$

where

$$A^2 = (\rho_2 - b)^2 + a^2 \qquad B^2 = (\rho_1 - b)^2 + a^2$$

$$k^2 = [(\rho_2 - \rho_1)^2 - (A - B)^2]/4AB \qquad \alpha^2 = -(A - B)^2/4AB$$

and where $\mathbf{K}(k)$ and $\mathbf{\Pi}(\alpha^2, k)$ are the complete elliptic integrals of the first and third

kinds respectively (Byrd and Friedman 1975), and

(ii) for $E > V_c$:

$$\frac{dE}{dn} = \hbar\omega \frac{\pi}{2}\left[\frac{2}{[(\rho_2 - c)(\rho_1 - d)]^{1/2}}\left(d\mathbf{K}(k) + (\rho_2 - d)\mathbf{\Pi}\left(\frac{\rho_1 - \rho_2}{\rho_1 - d}, k\right)\right)\right]^{-1} \tag{7}$$

where $k^2 = [(\rho_2 - \rho_1)(c - d)]/[(\rho_2 - c)(\rho_1 - d)]$.

Equations (6) and (7) give the spacing of the quasi-Landau resonances for any $T \geq 0$. For the particular case $T = 0$ they reduce to the previously reported results (Gallas and O'Connell 1982), as expected.

Before proceeding, we discuss in this paragraph the effect of the magnetic field upon the quantities V_c and V_0. When $T = 0$ it is clear from equation (1) that $V_0 \to -\infty$, independently of the applied magnetic field, and that $V_c = (27M\omega^2 e^4/32)^{1/3}$. When $T > 0$ it is always possible to make the difference $V_c - V_0$ arbitrarily small by increasing B sufficiently. However, as long as the Coulomb term is not neglected, $V_c - V_0$ will always remain greater than zero. For not too high T values, say $T < 15$, V_0 will be greater than zero whenever $B > B_0$, $B_0 \sim 10^6$ kG. For fields currently available in laboratories one has $B \ll B_0$ and, for fixed T, V_0 is practically independent of the magnetic field. The energy levels of the model described by equation (1) at very high B fields were recently discussed by Kara (1981) and Kara and McDowell (1981).

In numerical applications of equations (6) and (7) we calculated the elliptic integrals using the accurate and efficient algorithms developed by Carlson (1979). The positive roots ρ_1 and ρ_2 were calculated by the well known Newton method: using 10^{-2} and 10^4 as trial values for ρ_1 and ρ_2 respectively, the roots always converged within four or five iterations to 10^{-6} accuracy. From the polynomials in equations (4) and (5) it is easy to see that by defining $X = -(\rho_1 + \rho_2)/2$, $Y = \rho_1\rho_2 - 4/(X\omega^2) > 0$ and the discriminant $D = X^2 - Y$ it is always trivial to know if a given energy is above, below or at V_c. For $D > 0$, $E > V_c$ and the other two roots are $c = X + D^{1/2}$ and $d = X - D^{1/2}$. For $D \leq 0$, $E \leq V_c$ and it follows that $b = X$ and $a^2 = -D$. When $D = 0$, we are exactly at $E = V_c$. Since one always has to find ρ_1 and ρ_2, these last remarks considerably simplify the computation of dE/dn by avoiding the explicit need for determining V_c.

Using equations (6) and (7) above with $T = 1$, we computed and plotted the spacing for the same field values as were previously calculated numerically by Starace (1973) and plotted in his figure 1. To visual accuracy both figures are identical. The effect of the magnetic quantum number on the spacing can be seen in figure 2(*a*), where for $T = 0$ and $T = 100$ we have plotted the spacing as a function of the energy for the magnetic fields typically used in experiments. Since $T = 100$ gives $m = 10$ (assuming $T = m^2$), one sees that the effect is not very big. In particular, the use of $T = (m + \frac{1}{2})^2$ (Economou *et al* 1978), $T = m^2$ (Fonck *et al* 1980) or $T = (|m| + \frac{1}{2})^2$ (Gay *et al* 1980) for low m values and for fields between 10 and 50 kG will give essentially the same result. The quantitative difference can be inferred from figure 2(*b*). In this figure one also observes 'degeneracy' at the ionisation limit ($E = 0$): for $T = 0$ all the curves cross at 1.5 $\hbar\omega$ independently of the magnetic field. In figure 2(*c*) we show in even more detail the region where the $T = 100$ curves cross. This clearly shows that for $T \neq 0$ the above mentioned 'degeneracy' is 'removed' with every pair of curves crossing at different E and dE/dn values.

Figure 3 shows the effect of T, as a function of the magnetic field, on the spacings. It is interesting that the asymmetry with respect to the $dE/dn = 1.5\ \hbar\omega$ line observed in I can even be reversed for high enough T values.

Two cuts of the hypersurface $dE/dn = f(E, B, T)$ are shown in figure 4. For constant energy the T dependence becomes more important as B increases. For constant magnetic field the effect of T becomes more pronounced as the energy

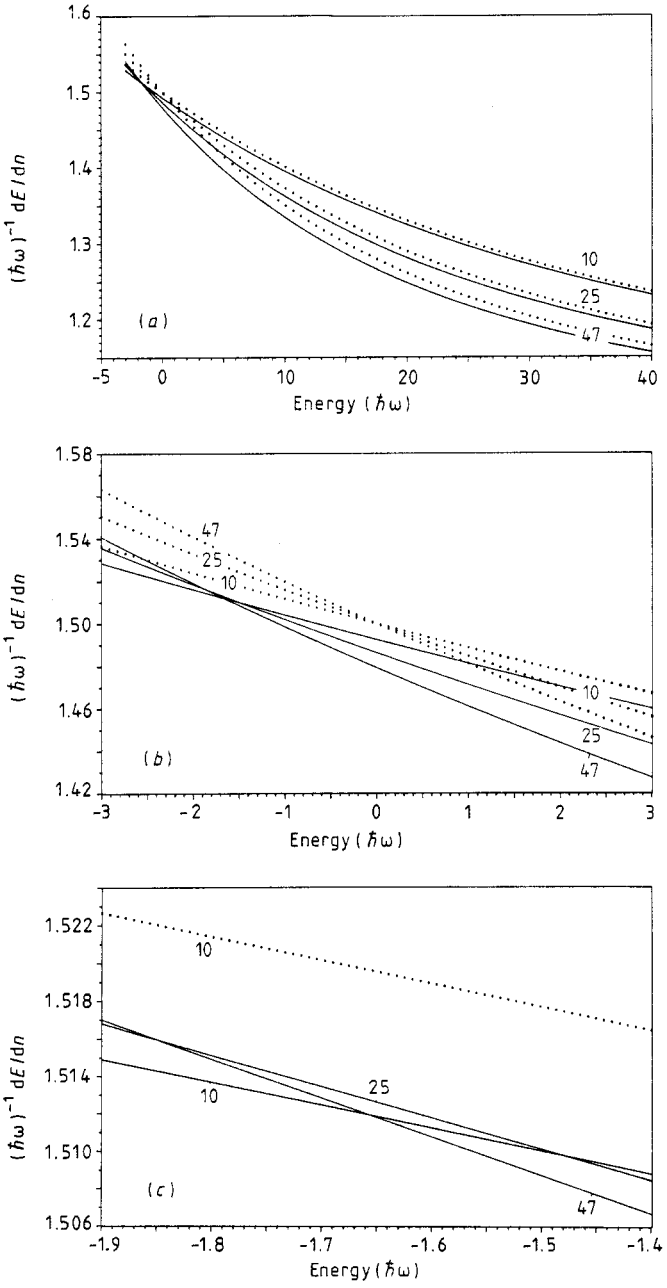

Figure 2. Plot of dE/dn against E for $T = 100$ (full curve) and $T = 0$ (dotted curve) for $B = 10$, 25 and 47 kG. (*a*) Behaviour above the ionisation limit ($E = 0$); (*b*) detail around the ionisation limit and (*c*) 'splitting' of the crossing at the ionisation limit for $T \neq 0$ (see text).

decreases. For values typically used in experiments, figure 5 gives more detailed cuts. Note the different vertical scales. Comparing the slopes of figures 5(*a*), (*b*) and (*c*), one sees that *the effect of T becomes more important for the combination B increasing*

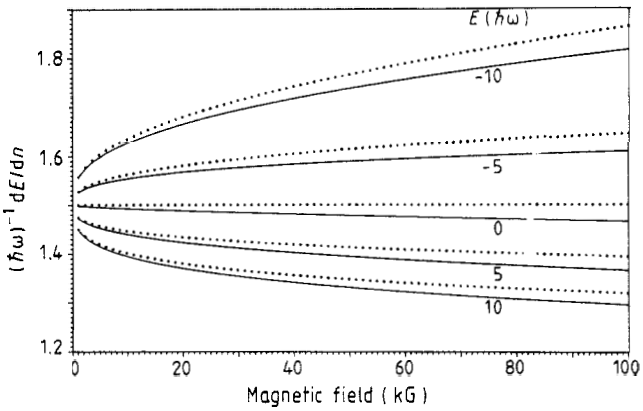

Figure 3. Spacing of the quasi-Landau resonances as a function of the magnetic field for $E = -10, -5, 0, 5$ and 10 in units of $\hbar\omega$ and for $T = 100$ (full curve) and $T = 0$ (dotted curve).

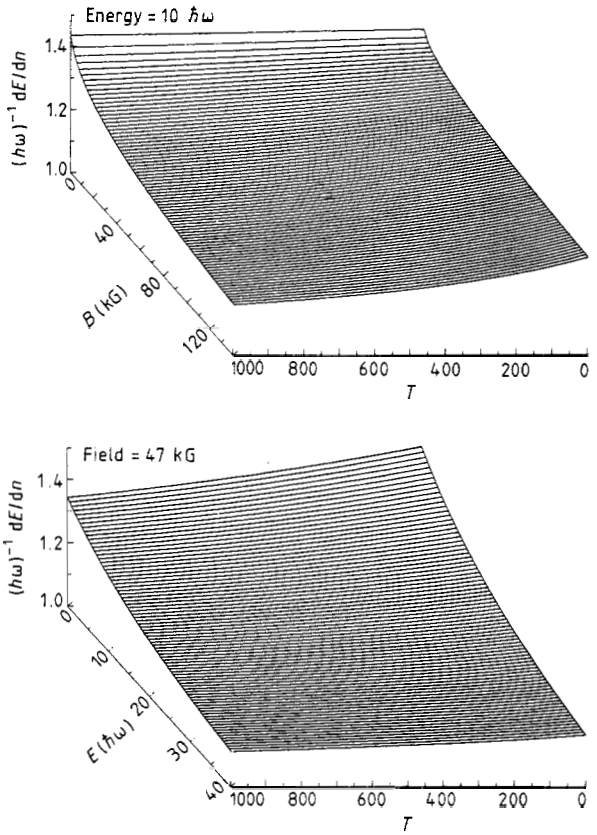

Figure 4. Two cuts of the hypersurface $dE/dn = f(E, B, T)$.

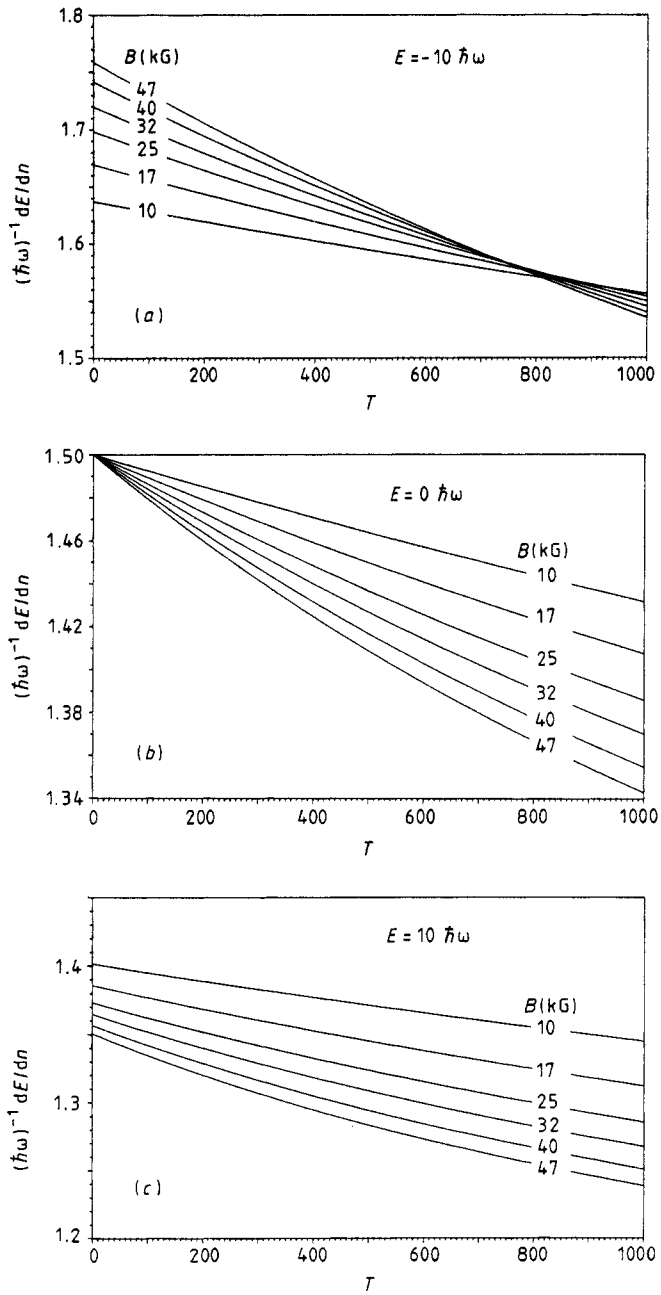

Figure 5. The spacing against T for $B = 10$, 17, 25, 32, 40 and 47 kG. (*a*) Below the ionisation limit; (*b*) at the ionisation limit and (*c*) above the ionisation limit.

and E decreasing. It is interesting to note the convergence around $T = 800$ ($m \simeq 28$) in figure 5(*a*).

As a final remark we observe that the quantisation rule itself (equation (2)) can be analytically integrated. This result will be reported elsewhere.

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